MTH 1125 - Test #2 - Solutions

Summer 2018

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Instructions:

Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[\tan\left(5x^3+8x^2\right)\right]=$

$$\begin{array}{rcl} \text{outer:} & = & \tan(\quad) \\ \text{deriv. of outer:} & = & \sec^2(\quad) \end{array}$$

$$\frac{d}{dx}\left[\underbrace{\tan}_{\text{outer}}\underbrace{\left(5x^3+8x^2\right)}_{\text{inner}}\right] = \underbrace{\sec^2\left(5x^3+8x^2\right)}_{\substack{\text{deriv. of outer}\\ \text{eval. at inner}}}\cdot\underbrace{\left(15x^2+16x\right)}_{\substack{\text{deriv. of}\\ \text{inner}}}$$

i.e., $\frac{d}{dx}\left[\tan\left(5x^3+8x^2\right)\right]=\sec^2\left(5x^3+8x^2\right)\left(15x^2+16x\right)$

2. Suppose that $x=\cos(t)$ and that $t=y^2-2y+3$. Compute $\frac{dx}{dy}$ **using the Liebniz form** of the Chain Rule.

We Know:

$$\begin{array}{rcl} \frac{dx}{dt} & = & -\sin(t) \\ \frac{dt}{dy} & = & 2y-2 \end{array}$$

We want: $\frac{dx}{dy}$

By the Liebniz form of the Chain Rule:

$$\frac{dx}{dy}=\frac{dx}{dt}\cdot\frac{dt}{dy}=-\sin(t)\cdot(2y-2)=-\sin\left(y^2-2y+3\right)\cdot(2y-2)$$

i.e., $\frac{dx}{dy}=-\sin\left(y^2-2y+3\right)\cdot(2y-2)$

3. Compute: $\frac{d}{dx}\left[\left(5x^6+7x^4+13x^2\right)^{15}\right]=$

$$\frac{d}{dx}\left[\left(5x^6+7x^4+13x^2\right)^{15}\right]=\frac{d}{dx}\left[\underbrace{\left(5x^6+7x^4+13x^2\right)^{15}}_{(g(x))^n}\right]=\underbrace{15\left(5x^6+7x^4+13x^2\right)^{14}}_{\substack{\text{power rule}\\ \text{as usual}}}\cdot\underbrace{\left(30x^5+28x^3+26x\right)}_{\substack{\text{deriv of}\\ \text{inner Function}}}$$

i.e., $\frac{d}{dx}\left[\left(5x^6+7x^4+13x^2\right)^{15}\right]=15\left(5x^6+7x^4+13x^2\right)^{14}\left(30x^5+28x^3+26x\right)$

4. Compute: $\int \left(5x^4 - 6x^2 + 4x + 5 + 3\sqrt{x}\right) dx =$

$$\int \left(5x^4 - 6x^2 + 4x + 5 + 3x^{\frac{1}{2}}\right) dx = 5\left[\frac{x^5}{5}\right] - 6\left[\frac{x^3}{3}\right] + 4\left[\frac{x^2}{2}\right] + 5x + 3\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right] + C$$

$$x^5 - 2x^3 + 2x^2 + 5x + 2x^{\frac{3}{2}} + C$$

> i.e., $\int \left(5x^4 - 6x^2 + 4x + 5 + 3\sqrt{x}\right) dx = x^5 - 2x^3 + 2x^2 + 5x + 2x^{\frac{3}{2}} + C$

5. Given that $8x^2 + 2x^3y^2 = 5y^3$; Compute y'

   i) Differentiate both sides with respect to x

   $$\Rightarrow \frac{d}{dx}\left[8x^2 + 2x^3y^2\right] = \frac{d}{dx}\left[5y^3\right]$$

   $$\Rightarrow 16x + \underbrace{6x^2 \cdot y^2 + 2y \cdot y' \cdot 2x^3}_{\text{product rule}} = \underbrace{15y^2}_{\substack{\text{Power Rule} \\ \text{as usual}}} \cdot \underbrace{y'}_{\text{deriv of inner}}$$

   i.e., $16x + 6x^2y^2 + 4yy'x^3 = 15y^2y'$

   ii) Solve for y' algebraically.

   a) Get the y' terms on the left side, all other term on the right side

   $$\Rightarrow 4yy'x^3 - 15y^2y' = -16x - 6x^2y^2$$

   b) Factor out y'

   $$\Rightarrow \left(4yx^3 - 15y^2\right) y' = -16x - 6x^2y^2$$

   c) Divide by the "cofactor" of y'

   $$\Rightarrow y' = \frac{-16x - 6x^2y^2}{4yx^3 - 15y^2} = -\frac{16x + 6x^2y^2}{4yx^3 - 15y^2}$$

   > i.e., $y' = \frac{-16x - 6x^2y^2}{4yx^3 - 15y^2} = -\frac{16x + 6x^2y^2}{4yx^3 - 15y^2}$

6. $f(x) = x^3 - 9x^2 + 24x + 2$

i) Determine the intervals on which $f(x)$ is increasing/decreasing

ii) Identify all relative maximums and minimums

i) Compute $f'(x)$ and find the critical numbers.

$f'(x) = 3x^2 - 18x + 24$

a) "Type a" $(f'(c) = 0)$

$\Rightarrow f'(x) = 3x^2 - 18x + 24 = 0$

$\Rightarrow 3x^2 - 18x + 24 = 0$

$\Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow (x-2)(x-4) = 0$

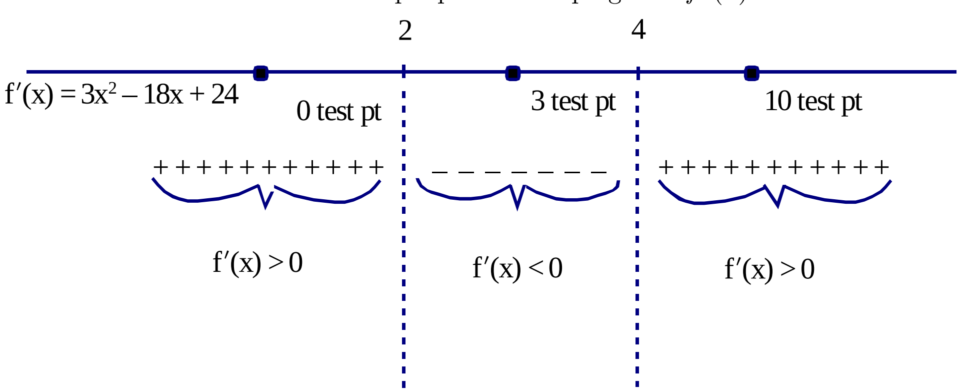
$\Rightarrow x = 2,\ x = 4$ are "type a" critical numbers

b) "Type b" ($f'(c)$ undefined)

There are none.

ii) Draw a "sign graph" of $f'(x)$

iii) From each interval. select a "sample point" and plug into $f'(x)$.



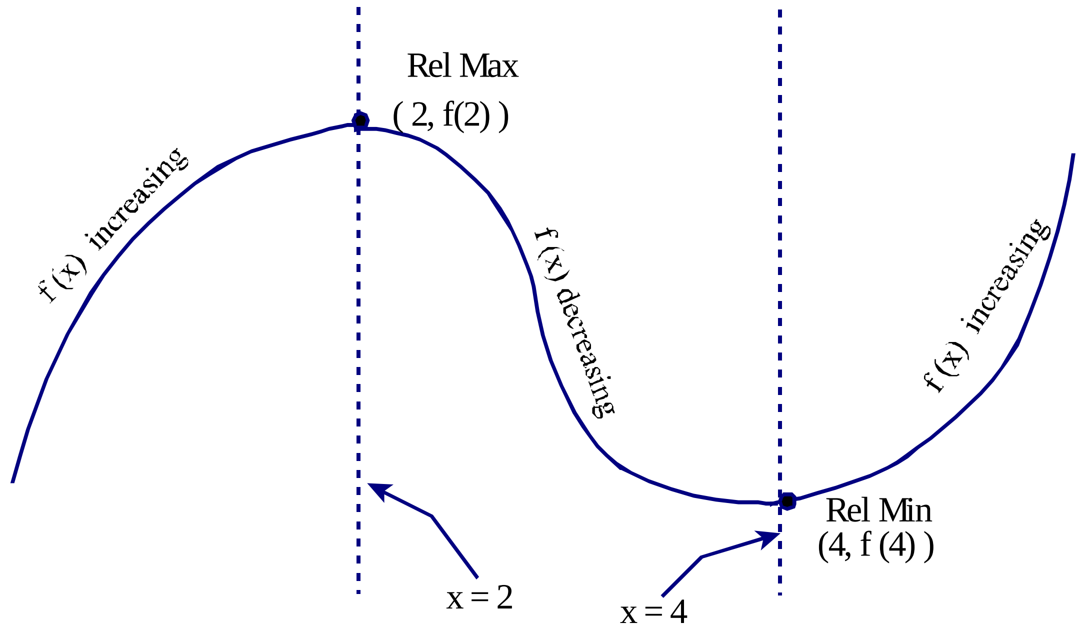
$f(x)$ is **increasing** on the intervals $(-\infty, 2)$ and $(4, \infty)$

(because $f'(x) > 0$ on these intervals).

$f(x)$ is **decreasing** on the interval $(2, 4)$

(because $f'(x) < 0$ on this interval).

iv) Sketch a rough graph of $f(x)$ to find the relative maxes and mins.



From the graph of $f(x) = x^3 - 9x^2 + 24x + 2$ it is clear that:

$(2, f(2)) = (2, 22)$ is a **relative max.**, and

$(4, f(4)) = (4, 18)$ is a **relative min**

7. $f(x) = 3x^{\frac{8}{3}} - 8x$

i) Determine the intervals on which $f(x)$ is increasing/decreasing

ii) Identify all relative maximums and minimums

i) Compute $f'(x)$ and find the critical numbers.

$f'(x) = 8x^{\frac{5}{3}} - 8$

a) "Type a" $(f'(c) = 0)$

$\Rightarrow f'(x) = 8x^{\frac{5}{3}} - 8 = 0$

$\Rightarrow x^{\frac{5}{3}} - 1 = 0$

$\Rightarrow x^{\frac{5}{3}} = 1$

$\Rightarrow x = 1$

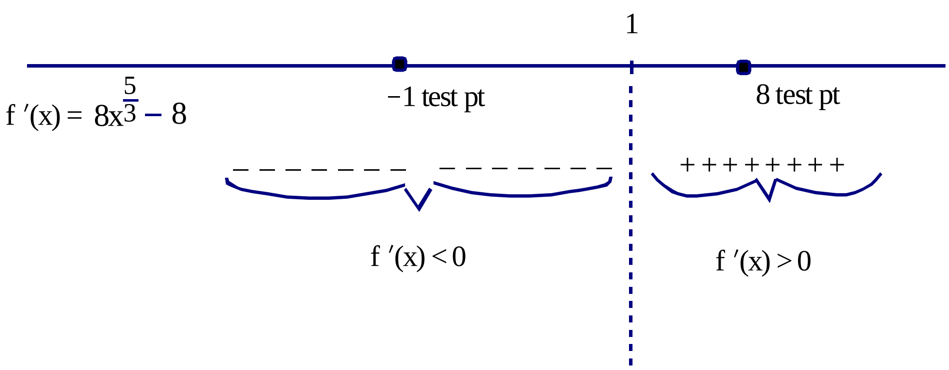
i.e., $x = 1$ is a "type a" critical number

b) "Type b" $(f'(c)$ undefined $)$

(None)

ii) Draw a "sign graph" of $f'(x)$

iii) From each interval. select a "sample point" and plug into $f'(x)$.



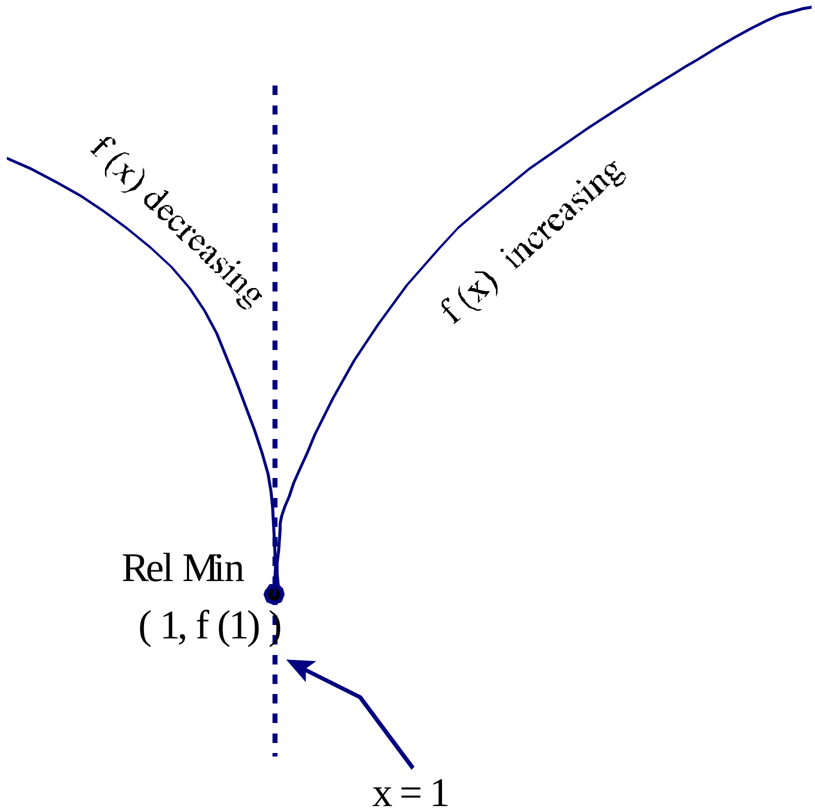
$f(x)$ is **increasing** on the interval $(1, \infty)$

(because $f'(x) > 0$ on this interval).

$f(x)$ is **decreasing** on the intervals $(-\infty, 1)$

(because $f'(x) < 0$ on these intervals).

iv) Sketch a rough graph of $f(x)$ to find the relative maxes and mins.



From the graph of $f(x) = 3x^{\frac{8}{3}} - 8x$ it is clear that:

$(1, f(1)) = (1, -5)$ is a **Relative Min.**

There is **NO Relative Max**

8. $f(x) = 2x^3 - 15x^2 + 36x + 6$ on the interval $[-2, 3]$. Find the absolute maximum and absolute minimum values.

Note: $f(x)$ is [1]continuous (no zero divides) on the [2]closed, [3]finite interval $[-3, 3]$.

Hence, we can use the Absolute Max/Min Value Test.

1. Find Critical numbers

   $f'(x) = 6x^2 - 30x + 36$

   a) (Type a)

      $f'(x) = 6x^2 - 30x + 36 = 0$

      $\Rightarrow x^2 - 5x + 6 = 0$

      $\Rightarrow (x-2)(x-3) = 0$

      $\Rightarrow x = 2$ and $x = 3$ are "type a" critical numbers

   b) (Type b)

      None

ii) Plug Critical numbers and endpoints into the *original function.*

   $f(-2) = 2(-2)^3 - 15(-2)^2 + 36(-2) + 6 = -142$ $\Leftarrow$ Abs. Min. Value

   $f(2) = 2(2)^3 - 15(2)^2 + 36(2) + 6 = 34$ $\Leftarrow$ Abs. Max. Value

   $f(3) = 2(3)^3 - 15(3)^2 + 36(3) + 6 = 33$

   The absolute maximum value is 34 (attained at $x = 2$)

   The absolute minimum value is -142 (attained at $x = -2$)

9. Compute: $\int \left(4x^3+3x^2\right)^{10}\left(12x^2+6x\right)dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(4x^3+3x^2\right)^{10}$

   $\nearrow$ inner, $\uparrow$ outer

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(4x^3+3x^2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(4x^3+3x^2\right)}_{\text{function}} - - - - - \rightarrow \underbrace{\left(12x^2+6x\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(4x^3+3x^2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 4x^3+3x^2 \\ \Rightarrow \frac{du}{dx} &= 12x^2+6x \\ \Rightarrow du &= \left(12x^2+6x\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(4x^3+3x^2\right)^{10}}_{u^{10}}\underbrace{\left(12x^2+6x\right)dx}_{du} = \int u^{10}\,du$$

4. Integrate (in terms of u).

$$\int u^{10}du = \left[\frac{u^{11}}{11}\right] + C = \frac{1}{11}u^{11} + C$$

5. Re-express in terms of the original variable, x.

$$\int \left(4x^3+3x^2\right)^{10}\left(12x^2+6x\right)dx = \underbrace{\frac{1}{11}\left(4x^3+3x^2\right)^{11} + C}_{\frac{1}{11}u^{11}+C}$$

i.e., $\int \left(4x^3+3x^2\right)^{10}\left(12x^2+6x\right)dx = \frac{1}{11}\left(4x^3+3x^2\right)^{11} + C$

10. $f(x) = x^4 - 12x^3 + 48x^2 + 6x + 6 =$

i) Determine the intervals on which $f(x)$ is concave up/concave down

ii) Identify all points of inflection

1. Compute $f''(x)$ and find the possible points of inflections.

   $f'(x) = 4x^3 - 36x^2 + 96x + 6$

   $f''(x) = 12x^2 - 72x + 96$

   a) "Type a" $(f''(c) = 0)$

      $\Rightarrow f''(x) = 12x^2 - 72x + 96 = 0$

      $\Rightarrow x^2 - 6x + 8 = 0$

      $\Rightarrow (x-2)(x-4) = 0$

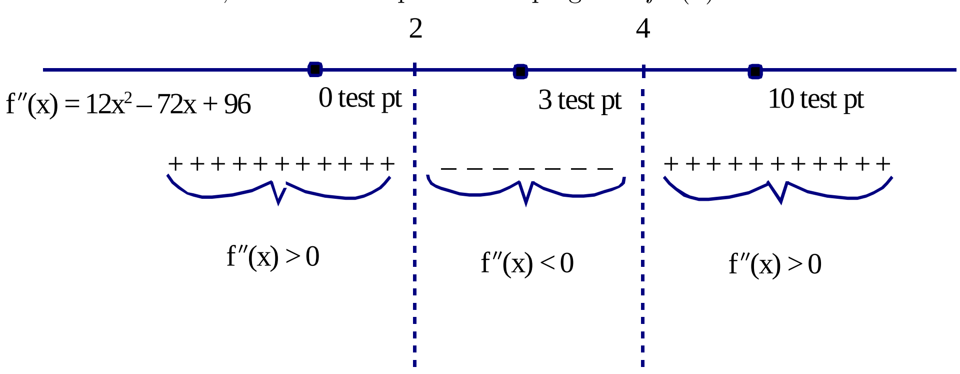
      $\Rightarrow x = 2,\ x = 4$ are possible "type a" points of inflection

   b) "Type b" $\left(f''(c) \text{ undefined}\right)$

      There are none.

2. Draw a "sign graph" of $f''(x)$

3. From each interval, select a "test point" and plug into $f''(x)$.



$f(x)$ is **concave up** on the intervals $(-\infty, 2)$ and $(4, \infty)$

(because $f''(x) > 0$ on these intervals).

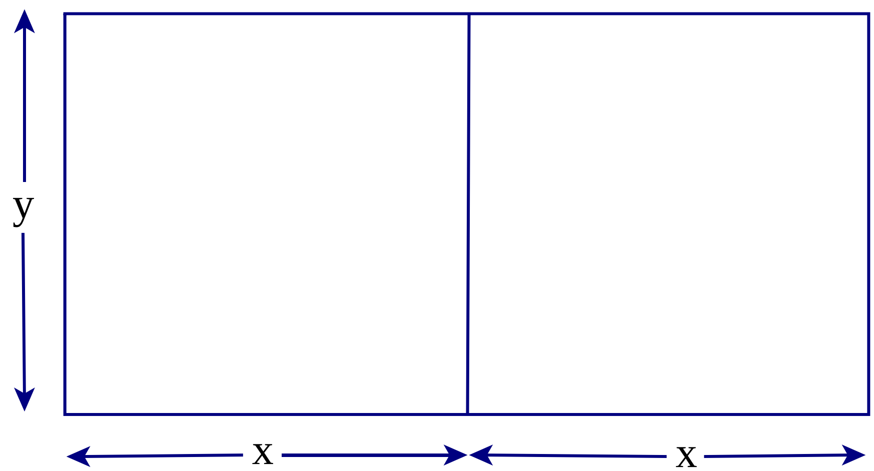
$f(x)$ is **concave down** on the interval $(2, 4)$

(because $f''(x) < 0$ on this interval).

Since $f(x)$ changes concavity at $x = 2$ and $x = 4$, the points

$(2, f(2)) = (2, 130)$ and $(4, f(4)) = (4, 286)$ are points of inflection.

11. Farmer Joe has 900 ft of fence. He will use the fence to construct a rectangular pen. He will use some of the fence to partition the pen into two smaller pens of equal area and similar shape. What should the dimensions of the pen be so that the overall enclosed area is as large as possible?



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = 2xy$

   a) Draw a picture where relevant.

      (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that Farmer Joe must use exactly 1000 ft of fence.

   $\Rightarrow 4x + 3y = 900$ ft

   $\Rightarrow 3y = 900$ ft $-4x$

   $\Rightarrow y = 300$ ft $-\frac{4}{3}x$

   Plug this into the equation $A = 2xy$

   $\Rightarrow A(x) = 2x\left(300 \text{ ft } - \frac{4}{3}x\right) = 600 \text{ ft } x - \frac{8}{3}x^2$

   i.e., $A(x) = 600 \text{ ft } x - \frac{8}{3}x^2$

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \text{ ft} \le x \le 225 \text{ ft}$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0 \text{ ft}, 225 \text{ ft}]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = 600 \text{ ft} - \frac{8}{3}x$

      a. "Type a" $(A'(c) = 0)$

         $\Rightarrow A'(x) = 600 \text{ ft} - \frac{16}{3}x = 0$

         $\Rightarrow \frac{16}{3}x = 600 \text{ ft}$
         $\Rightarrow x = \frac{1800}{16} \text{ ft} = \frac{225}{2} \text{ ft}$ is a critical number

      b. "Type b" $(A'(c)$ is undefined)

         Look for x-values that cause division by zero in $A'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0 \text{ ft}) = 600 \text{ ft}\,(0 \text{ ft}) - \frac{8}{3}(0 \text{ ft})^2 = 0 \text{ ft}^2$

      $A\left(\frac{225}{2} \text{ ft}\right) = 600 \text{ ft}\left(\frac{225}{2} \text{ ft}\right) - \frac{8}{3}\left(\frac{225}{2} \text{ ft}\right)^2 = 33,750 \text{ ft}^2 \longleftarrow$ Abs Max Value

      $A(225 \text{ ft}) = 600 \text{ ft}\,(225 \text{ ft}) - \frac{8}{3}(225 \text{ ft})^2 = 0 \text{ ft}^2$

5. Make sure that we've answered the original question.

   "What should the dimensions of the pen be so that the overall enclosed area is as large as possible?"

   Length $= 2x = 2\left(\frac{225}{2} \text{ ft}\right) = 225 \text{ ft}$

   Width $= y = \frac{900}{3} - \frac{4}{3}x = y = \frac{900}{3} \text{ ft} - \frac{4}{3}\left(\frac{225}{2} \text{ ft}\right) = 150 \text{ ft}$

Length $= 2x = 225$ ft
Width $= y = 150$ ft

12. $f = \begin{cases} x^2 & \text{for } x < 2 \\ 4 & \text{for } x = 2 \\ \frac{x^2-2x}{x-2} & \text{for } x > 2 \end{cases}$ Determine whether the function $f(x)$ is continuous at the point $x = 2$

If $f(x)$ **is** continuous at the point $x = 2$, then $\lim_{x\to 2} f(x) = f(2)$.

To see if this is true, we'll compute $\lim_{x\to 2} f(x)$.

Since the definition of $f(x)$ for $x < 2$ is different than the definition of $f(x)$ for $x > 2$, we must compute the one-sided limits in order to determine whether the limit exists.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = (2)^2 = 4$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2-2x}{x-2} = \lim_{x\to 2^+} \frac{(x-2)x}{x-2} = \lim_{x\to 2^+} x = 2$

Since the one-sided limits are NOT EQUAL, $\lim_{x\to 2} f(x)$ does NOT exist.

Hence, $\lim_{x\to 2} f(x) \neq f(2)$,

$f(x)$ is **NOT** continuous at $x = 2$