MTH 1125 (9 am) Test #3 – Solutions

FALL 2017

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Show CLEARLY how you arrive at your answers.

1. Given that $y^3 + x^3 = x^3y^3$; compute y'

   i. Differentiate both sides w.r.t. x

   $\frac{d}{dx}\left[y^3 + x^3\right] = \frac{d}{dx}\left[\underbrace{x^3}_{1^{\text{st}}}\underbrace{y^3}_{2^{\text{nd}}}\right]$

   $\Rightarrow 3y^2 \cdot y' + 3x^2 = \underbrace{3x^2}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{y^3}_{2^{\text{nd}}} + \underbrace{3y^2 \cdot y'}_{2^{\text{nd}}\text{prime}} \cdot \underbrace{x^3}_{1^{\text{st}}}$

   ii. Solve algebraically for y'

      a. Get y' terms on left side, all other terms on right side

      $\Rightarrow 3y^2y' - 3y^2y'x^3 = 3x^2y^3 - 3x^2$

      b. Factor out y'

      $\Rightarrow \left(3y^2 - 3y^2x^3\right)y' = 3x^2y^3 - 3x^2$

      c. Divide both sides by the cofactor of y'

      $y' = \frac{3x^2y^3 - 3x^2}{3y^2 - 3y^2x^3}$

$$\boxed{y' = \frac{3x^2y^3 - 3x^2}{3y^2 - 3y^2x^3} = \frac{x^2y^3 - x^2}{y^2 - y^2x^3}}$$

2. $f(x) = x^3 - 3x^2 - 24x + 2$. [1]Identify the intervals on which $f(x)$ is increasing/decreasing, and [2]identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find critical numbers

   $f'(x) = 3x^2 - 6x - 24$

      a. "Type a" $(f'(c) = 0)$

      Set $f'(x) = 3x^2 - 6x - 24 = 0$

      $\Rightarrow 3x^2 - 6x - 24 = 0$

      $\Rightarrow x^2 - 2x - 8 = 0$

      $\Rightarrow (x-4)(x+2) = 0$

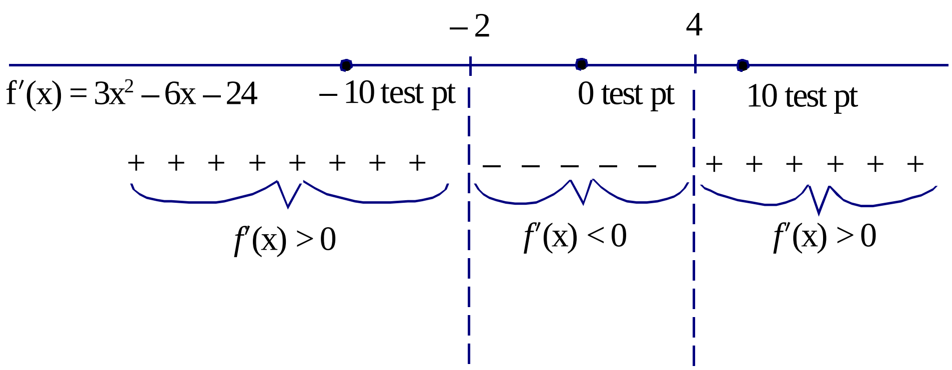
      $\Rightarrow x = -2;\ x = 4$ critical numbers

      b. "Type b" $(f'(c)$ undefined$)$

      There are none.

   ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

   iii. From each interval select a "test point" to plug into $f'(x)$



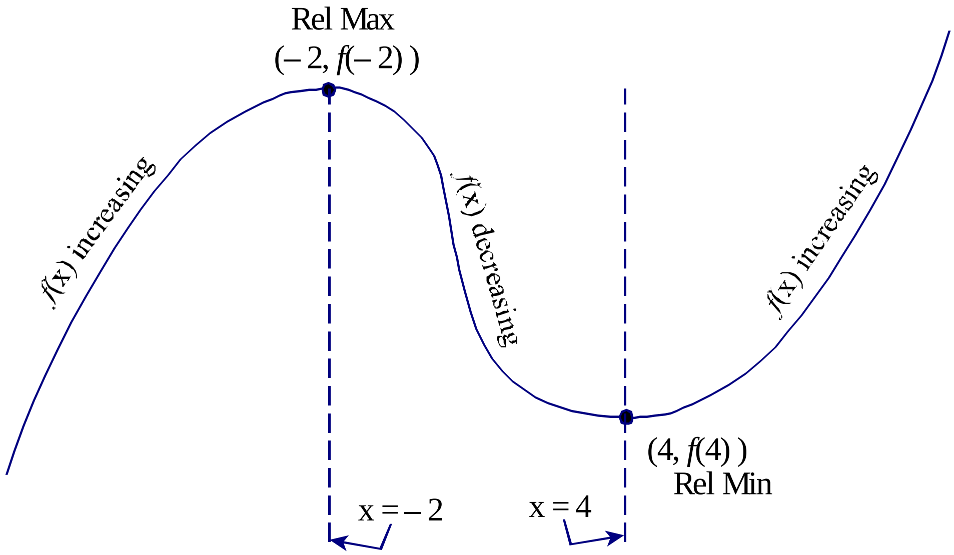
$f(x)$ is **increasing** on the intervals $(-\infty, -2)$ and $(4, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval $(-2, 4)$

(Because $f'(x)$ is negative on this interval)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



$f(x)$ is **increasing** on the intervals $(-\infty, -2)$ and $(4, \infty)$

(Because $f'(x)$ is **negative)**

$f(x)$ is **decreasing** on the interval $(-2, 4)$

(Because $f'(x)$ is **positive)**

$(-2, f(-2)) = (-2, 30)$ Relative Max

$(4, f(4)) = (4, -78)$ Relative Min

3. $f(x) = 4x^{\frac{5}{3}} - 10x^{\frac{2}{3}} + 1$. [1]Identify the intervals on which $f(x)$ is increasing/decreasing, and [2]identify all relative maximums and minimums.

i. Compute $f'(x)$ and find critical numbers

$f'(x) = \frac{20}{3}x^{\frac{2}{3}} - \frac{20}{3}x^{-\frac{1}{3}} = \frac{20x^{\frac{2}{3}}}{3} - \frac{20}{3x^{\frac{1}{3}}} = \frac{20x^{\frac{2}{3}}}{3}\frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}} - \frac{20}{3x^{\frac{1}{3}}} = \frac{20x-20}{3x^{\frac{1}{3}}}$

i.e., $f'(x) = \frac{20x-20}{3x^{\frac{1}{3}}}$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = \frac{20x-20}{3x^{\frac{1}{3}}} = 0$

$\Rightarrow 20x - 20 = 0$

$\Rightarrow x - 1 = 0$

$\Rightarrow x = 1$ critical number

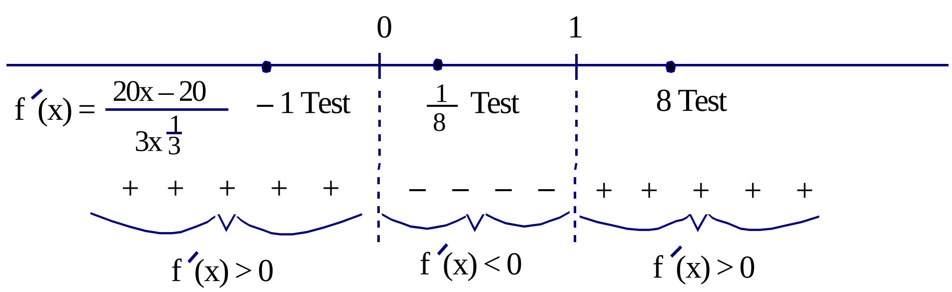
b. "Type b" $(f'(c)$ undefined$)$

Set denominator $3x^{\frac{1}{3}} = 0$

$\Rightarrow x = 0$ critical number

ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

iii. From each interval select a "test point" to plug into $f'(x)$



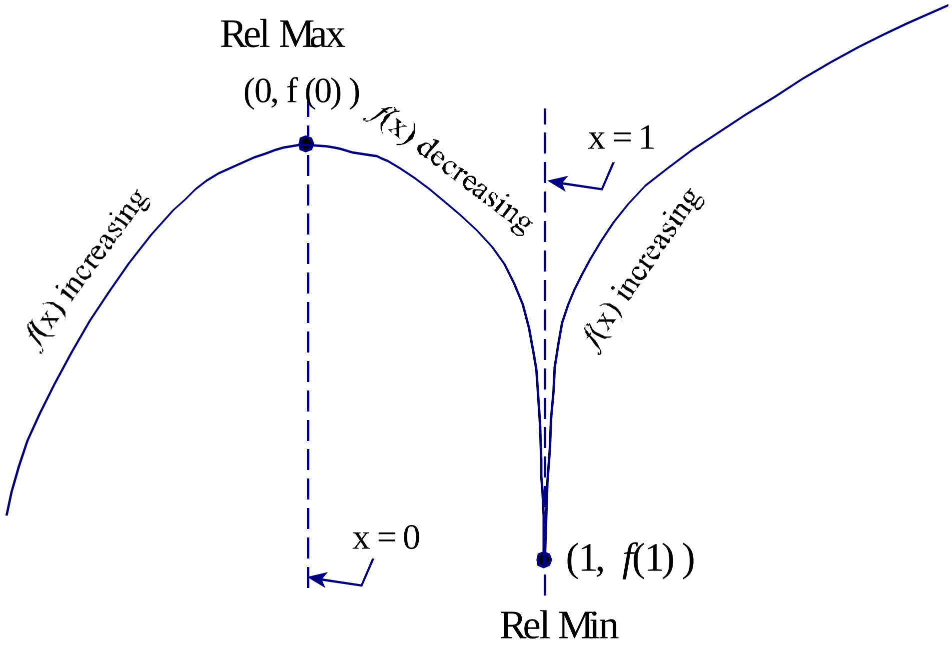
$f(x)$ is **increasing** on the intervals $(-\infty, 0)$ and $(1, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the intervals $(0, 1)$

(Because $f'(x)$ is negative on this interval.

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



$f(x)$ is **increasing** on the intervals $(-\infty, 0)$ and $(1, \infty)$

$f(x)$ is **decreasing** on the interval $(0, 1)$

Relative Max $(0, f(0)) = (0, 1)$

Relative Min $(1, f(1)) = (0, -5)$

4. $f(x) = 4x^3 - 3x^2 - 6x + 2$ on the interval $[0, 2]$. Find the absolute maximum value and absolute minimum value of $f(x)$.

Since $f(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 3]$, we can use the Absolute Max/Min Value Test

i. Compute $f'(x)$ and find the critical numbers

$f'(x) = 12x^2 - 6x - 6$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = 12x^2 - 6x - 6 = 0$

$\Rightarrow 12x^2 - 6x - 6 = 0$

$\Rightarrow 2x^2 - x - 1 = 0$

$\Rightarrow (2x + 1)(x - 1) = 0$

$\Rightarrow x = -\frac{1}{2}$; $x = 1$ "type a" crit. numbers

Alternatively: To find the critical numbers, we take the equation $2x^2 - 3x - 2 = 0$ and plug the coefficients into the quadratic equation:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2(2)} = \frac{1 \pm \sqrt{9}}{4}$

$\Rightarrow x = -\frac{1}{2}$; $x = 1$ "type a" crit. numbers

Since $x = -\frac{1}{2}$ is not in the interval $[0, 2]$, we discard $x = -\frac{1}{2}$ as a critical number.

b. "Type b" ($f'(c)$ is undefined)

No "type b" critical numbers.

ii. Plug critical numbers and endpoints into the original function.

$f(0) = 4(0)^3 - 3(0)^2 - 6(0) + 2 = 2$

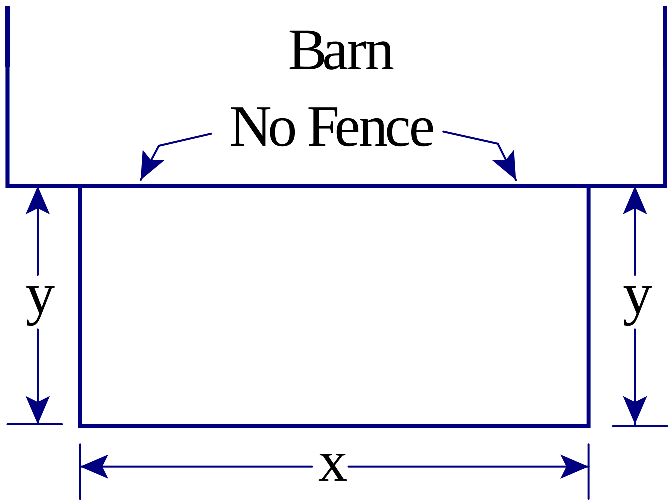
$f(1) = 4(1)^3 - 3(1)^2 - 6(1) + 2 = -3 \leftarrow$ Abs MinValue

$f(2) = 4(2)^3 - 3(2)^2 - 6(2) + 2 = 10 \leftarrow$ Abs MaxValue

Abs Max Value $= 10$
(attained at $x = 2$)

Abs Min Value $= -3$
(attained at $x = 1$)

5. Farmer Joe has 100 feet of wire fence. He will use it to construct a rectangular pen. If the side of a barn will form one side of the pen, what should the dimensions of the pen be in order for the pen to enclose as large an area as possible? (No fence will be used on the side of the pen that borders the barn.)



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the box, $A = xy$

   a. Draw a picture where relevant.

   (Done)

2. Express A as a function of one other variable.

   (Typically, we refer to a restriction, stated in the exercise, in order to do this.)

   Restriction: Joe must use exactly 200 ft of fencing.

   Joe will use this fencing to form the side of length x ans to form the two sides of length y.

   Thus, $x + 2y = 100$ ft

   $\Rightarrow 2y = 100 \text{ ft} - x$

   $\Rightarrow y = 50 \text{ ft} - \frac{1}{2}x$

   Plug this expression for y into the equation $A = xy$

   $\Rightarrow A = x\left(50 \text{ ft} - \frac{1}{2}x\right) = 50 \text{ ft } x - \frac{1}{2}x^2$

   i.e., $A\left(x\right) = 50 \text{ ft } x - \frac{1}{2}x^2$

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \text{ ft} \leq x \leq 100 \text{ ft}$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0 \text{ ft}, 100 \text{ ft}]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = 50\text{ft} - x$

      a. "Type a" $(A'(c) = 0)$

         $\Rightarrow A'(x) = 50\text{ft} - x = 0$

         $\Rightarrow 50\text{ft} - x = 0$

         $\Rightarrow x = 50 \text{ ft}$

      b. "Type b" ($A'(c)$ is undefined)

         Look for x-values that cause division by zero in $A'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0\text{ft}) = 50\text{ft}\ (0\text{ft}) - \frac{1}{2}(0\text{ft})^2 = 0\text{ft}^2$

      $A(50\text{ft}) = 50\text{ft}\ (50\text{ft}) - \frac{1}{2}(50\text{ft})^2 = 1,250\text{ft}^2 \longleftarrow$ Abs Max Value

      $A(100\text{ft}) = 50\text{ft}\ (100\text{ft}) - \frac{1}{2}(100\text{ft})^2 = 0\text{ft}^2$

5. Make sure that we've answered the original question.

   ". . . what should the dimensions of the pen be in order for the pen to enclose as large an area as possible?"

   The area is as large as possible when

   Length $x = 50\text{ft}$

   Height $y = 50 \text{ ft} - \frac{1}{2}x = 50 \text{ ft} - \frac{1}{2}(50\text{ft}) = 25 \text{ ft}$

Length $= x = 50$ ft
Width $= y = 25$ ft