

MTH 1125 (1 pm) Test #3 – Solutions

FALL 2018

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Name _____

Show CLEARLY how you arrive at your answers.

1. $f(x) = 2x^3 - 3x^2 - 36x + 1$. ¹Identify the intervals on which $f(x)$ is increasing/decreasing, and ²identify all relative maximums and minimums.

- i. Compute $f'(x)$ and find critical numbers

$$f'(x) = 6x^2 - 6x - 36$$

- a. “Type a” ($f'(c) = 0$)

$$\text{Set } f'(x) = 6x^2 - 6x - 36 = 0$$

$$\Rightarrow 6x^2 - 6x - 36 = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x + 2)(x - 3) = 0$$

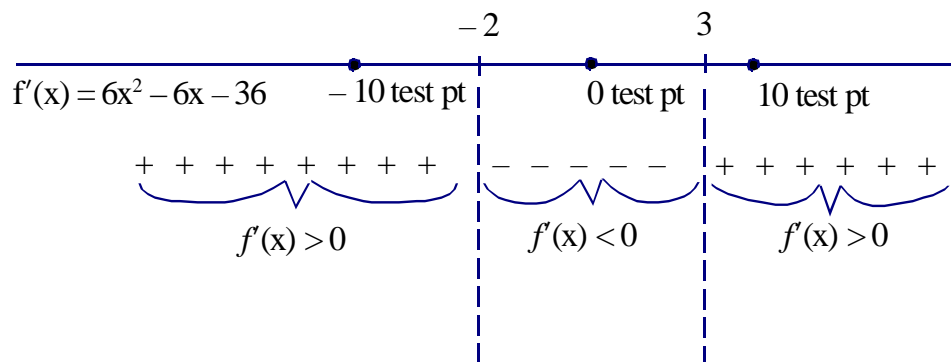
$$\Rightarrow x = -2; x = 3 \text{ critical numbers}$$

- b. “Type b” ($f'(c)$ undefined)

There are none.

- ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x -axis

- iii. From each interval select a “test point” to plug into $f'(x)$



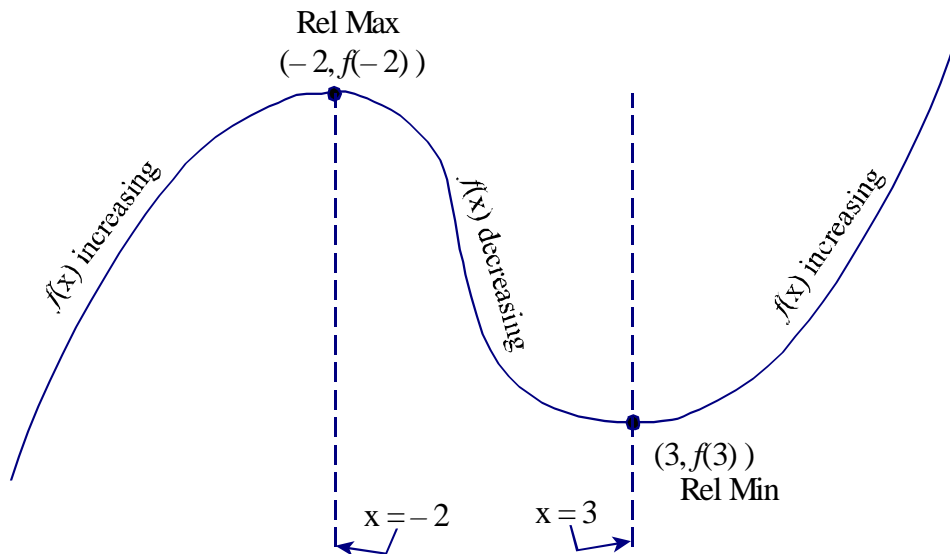
$f(x)$ is **increasing** on the intervals $(-\infty, -2)$ and $(3, \infty)$

(Because $f'(x)$ is **positive** on these intervals)

$f(x)$ is **decreasing** on the interval $(-2, 3)$

(Because $f'(x)$ is **negative** on this interval)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



$f(x)$ is **increasing** on the intervals $(-\infty, -2)$ and $(3, \infty)$

(Because $f'(x)$ is **positive**)

$f(x)$ is **decreasing** on the interval $(-2, 3)$

(Because $f'(x)$ is **negative**)

Relative Max $(-2, f(-2)) = (-2, 45)$

Relative Min $(3, f(3)) = (3, -80)$

2. $f(x) = \frac{3}{22}x^{\frac{11}{3}} - \frac{24}{5}x^{\frac{5}{3}} + 4x + 2$. ¹Identify the intervals on which $f(x)$ is concave up/concave down, and ²identify all points of inflection

i. Compute $f'(x)$ and find critical numbers

$$f'(x) = \frac{1}{2}x^{\frac{8}{3}} - 8x^{\frac{2}{3}}$$

$$f''(x) = \frac{4}{3}x^{\frac{5}{3}} - \frac{16}{3}x^{-\frac{1}{3}} = \frac{4x^{\frac{5}{3}}}{3} - \frac{16}{3x^{\frac{1}{3}}} = \frac{4x^{\frac{5}{3}}}{3} \cdot \frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}} - \frac{16}{3x^{\frac{1}{3}}} = \frac{4x^2}{3x^{\frac{1}{3}}} - \frac{16}{3x^{\frac{1}{3}}} = \frac{4x^2-16}{3x^{\frac{1}{3}}}$$

$$\text{i.e., } f''(x) = \frac{4x^2-16}{3x^{\frac{1}{3}}}$$

Find the possible points of inflection

a. "Type a" ($f''(c) = 0$)

$$\text{Set } f''(x) = \frac{4x^2-16}{3x^{\frac{1}{3}}} = 0$$

$$\Rightarrow 4x^2 - 16 = 0$$

$$\Rightarrow x^2 - 4 = 0$$

$$\Rightarrow x^2 - 4 = 0$$

$$\Rightarrow (x+2)(x-2) = 0$$

$$\Rightarrow x = -2; x = 2 \quad \text{possible Points of Inflection}$$

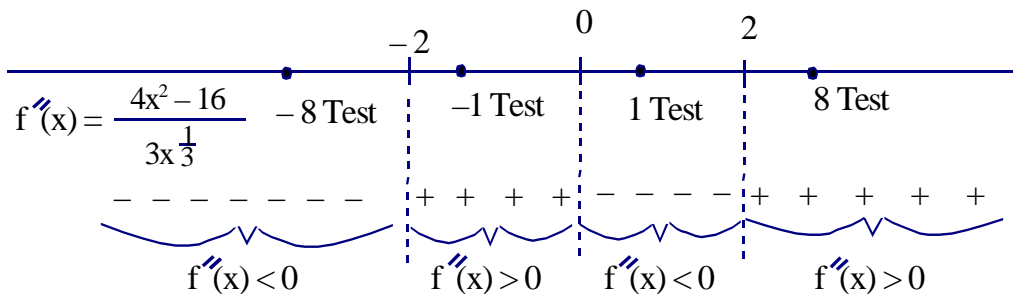
b. "Type b" ($f'(c)$ undefined)

$$\text{Set denominator } 3x^{\frac{1}{3}} = 0$$

$$\Rightarrow x = 0 \quad \text{possible Point of Inflection}$$

ii. Draw a sign graph of $f''(x)$, using the possible points of inflection to partition the x -axis

iii. From each interval select a "test point" to plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-2, 0)$ and $(2, \infty)$

(Because $f''(x)$ is **positive** on these intervals)

$f(x)$ is **concave down** on the intervals $(-\infty, -2)$ and $(0, 2)$

(Because $f''(x)$ is **negative** on this interval.)

Since $f''(x)$ changes sign at $x = -2$, $x = 0$, and $x = 2$, **the points:**

$(-2, f(-2))$; $(0, f(0))$; and $(2, f(2))$ **ARE points of inflection.**

3. $f(x) = 4x^3 + 3x^2 - 6x + 2$ on the interval $[-2, 0]$. Find the absolute maximum value and absolute minimum value of $f(x)$.

Since $f(x)$ is ¹continuous (it's a polynomial) on the ²closed, ³finite interval $[0, 3]$, we can use the Absolute Max/Min Value Test

- i. Compute $f'(x)$ and find the critical numbers

$$f'(x) = 12x^2 + 6x - 6$$

- a. "Type a" ($f'(c) = 0$)

$$\text{Set } f'(x) = 12x^2 + 6x - 6 = 0$$

$$\Rightarrow 12x^2 + 6x - 6 = 0$$

$$\Rightarrow 2x^2 + x - 1 = 0$$

$$\Rightarrow (2x - 1)(x + 1) = 0$$

$$\Rightarrow x = -1; x = \frac{1}{2} \text{ "type a" crit. numbers}$$

Alternatively: To find the critical numbers, we take the equation $2x^2 + x - 1 = 0$ and plug the coefficients into the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{(1)^2 - 4(2)(-1)}}{2(2)} = \frac{-1 \pm \sqrt{9}}{4}$$

$$\Rightarrow x = -1; x = \frac{1}{2} \text{ "type a" crit. numbers}$$

Since $x = \frac{1}{2}$ is not in the interval $[-2, 0]$, we discard $x = \frac{1}{2}$ as a critical number.

- b. "Type b" ($f'(c)$ is undefined)

No "type b" critical numbers.

- ii. Plug critical numbers and endpoints into the original function. $4x^3 + 3x^2 - 6x + 2$

$$f(-2) = 4(-2)^3 + 3(-2)^2 - 6(-2) + 2 = -6 \leftarrow \text{Abs Min Value}$$

$$f(-1) = 4(-1)^3 + 3(-1)^2 - 6(-1) + 2 = 7 \leftarrow \text{Abs Max Value}$$

$$f(0) = 4(0)^3 + 3(0)^2 - 6(0) + 2 = 2$$

Abs Max Value = 7 (attained at $x = -1$)
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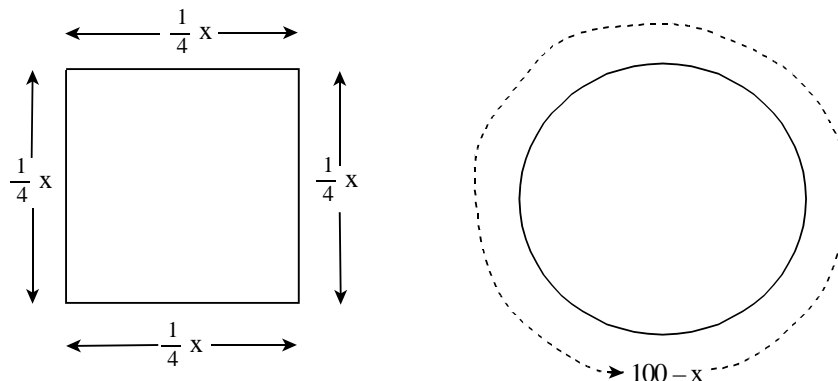
Abs Min Value = -6 (attained at $x = -2$)

4. Farmer Joe has 100 yards of fencing. He will use some of the fencing to construct a square pen and he will use the rest of the fencing to construct a circular pen. How much fencing should he use to construct each if he wants the total enclosed area to be as large as possible?

Let x be the amount of fencing that he will use on the square pen

Then $(100 - x)$ is the amount of fencing that he will use on the circular pen

Then the circumference of the circular pen will be $C = 2\pi r = 100 - x$



Let A be the total area enclosed.

Then $A = (\text{area of the square}) + (\text{area of the circle}) = \left(\frac{x}{4}\right)^2 + \pi r^2$

$$A = \left(\frac{x}{4}\right)^2 + \pi r^2$$

We need to know what r is in terms of x

Recall that the circumference is $C = 2\pi r = 100 - x$

$$\Rightarrow 2\pi r = 100 - x$$

$$\Rightarrow r = \frac{100-x}{2\pi}$$

Thus, Area is given by:

$$A(x) = \left(\frac{x}{4}\right)^2 + \pi r^2 = \left(\frac{x}{4}\right)^2 + \pi \left(\frac{100-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{1}{4\pi} (100 - x)^2$$

$$\text{i.e., } A(x) = \frac{x^2}{16} + \frac{1}{4\pi} (100 - x)^2$$

Again, we want to find out: How much fencing should be used to construct each pen, if the total enclosed area to be as large as possible?

(Use $\pi = \frac{22}{7}$ in your final calculations)

3. Determine the restrictions on x

$$0 \text{ yards} \leq x \leq 100 \text{ yards}$$

4. Maximize the area, using the techniques of calculus.

Note that since $A(x)$ is ¹continuous on the ²closed, ³finite interval $[0 \text{ yards}, 100 \text{ yards}]$, we can use the Absolute Max/Min Value Test.

i) Compute $A'(x)$ and find the critical numbers

$$A'(x) = \frac{1}{8}x + \frac{1}{2\pi}(100 - x)(-1) = \frac{1}{8}x + \frac{1}{2\pi}(x - 100) = \left(\frac{1}{8} + \frac{1}{2\pi}\right)x - \frac{100}{2\pi} = \frac{\pi+4}{8\pi}x - \frac{100}{2\pi}$$

$$\text{i.e., } A'(x) = \frac{\pi+4}{8\pi}x - \frac{100}{2\pi}$$

Find critical numbers

a) "Type a" ($A'(x) = 0$)

$$A'(x) = \frac{\pi+4}{8\pi}x - \frac{100}{2\pi} = 0$$

$$\Rightarrow \frac{\pi+4}{8\pi}x = \frac{100}{2\pi}$$

$$\Rightarrow \frac{\pi+4}{4}x = 100$$

$$\Rightarrow x = \frac{400}{\pi+4} = \frac{400}{\left(\frac{22}{7}\right)+4} = \frac{400}{\frac{22}{7}+\frac{28}{7}} = \frac{400}{\left(\frac{50}{7}\right)} = \frac{7}{50} \frac{400}{1} = 56$$

i.e., $x = 56$ critical number

b) "Type b" ($A'(x)$ undefined)

(None)

ii) Plug critical numbers and endpoints into the **original** function

$$A(0 \text{ yds}) = \frac{(0 \text{ yds})^2}{16} + \frac{1}{4\pi}(100 \text{ yds} - (0 \text{ yds}))^2 = \frac{2500}{\pi} \text{ yds}^2 \approx \frac{2500}{\left(\frac{22}{7}\right)} \text{ yds}^2 = \frac{7}{22} \frac{2500}{1} \text{ yds}^2$$

$$= \frac{8750}{11} \text{ yds}^2 = \left(795 + \frac{5}{11}\right) \text{ yds}^2 \leftarrow \text{Absolute Maximum Area}$$

$$A(56 \text{ yds}) = \frac{(56)^2}{16} + \frac{1}{4\pi}(100 - (56))^2 = 196 + \frac{1}{4\pi}(44)^2 \approx 196 + \frac{1}{4\left(\frac{22}{7}\right)}(44)^2 = 196 + \frac{7}{88}(44)^2$$

$$= 196 + (7)(22) = 350$$

$$A(100 \text{ yds}) = \frac{(100)^2}{16} + \frac{1}{4\pi}(100 - (100))^2 = 625 \text{ yds}^2$$

5. Make sure that we've answered the original question

"How much fencing should he use to construct each . . ."

Since $x = 0$ yds yields the maximum total area, and since x is the amount of fencing that he will use on the square pen, **He should use 0 yards of fencing on the square pen and 100 yards of fencing on the circular pen.**