MTH 1125 (2 pm) Test #3 – Solutions

Fall 2018

Pat Rossi Name ______________________

Show CLEARLY how you arrive at your answers.

1. $f(x) = x^3 - 3x^2 + 6$. [1]Identify the intervals on which $f(x)$ is increasing/decreasing, and [2]identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find critical numbers

   $f'(x) = 3x^2 - 6x$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 3x^2 - 6x = 0$

   $\Rightarrow 3x^2 - 6x = 0$

   $\Rightarrow x^2 - 2x = 0$

   $\Rightarrow x(x-2) = 0$

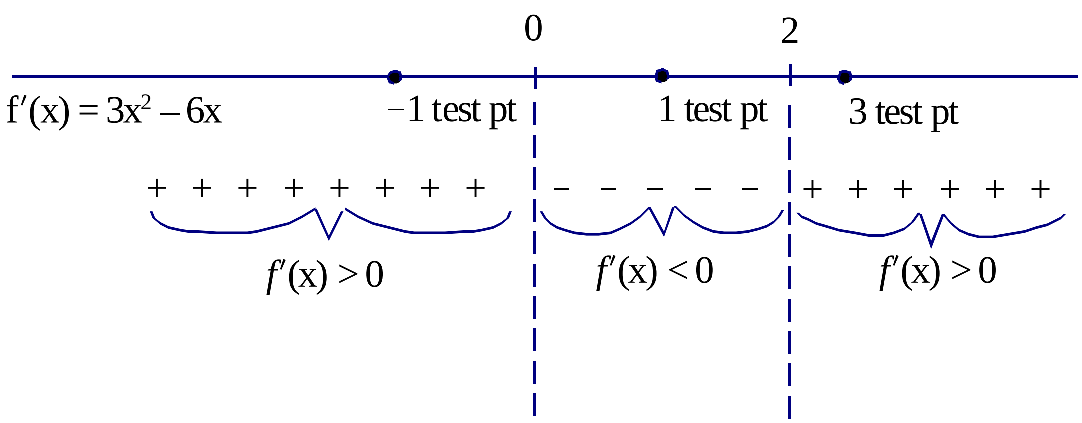
   $\Rightarrow x = 0$; $x = 2$ critical numbers

   b. "Type b" $(f'(c)$ undefined$)$

   There are none.

   ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

   iii. From each interval select a "test point" to plug into $f'(x)$



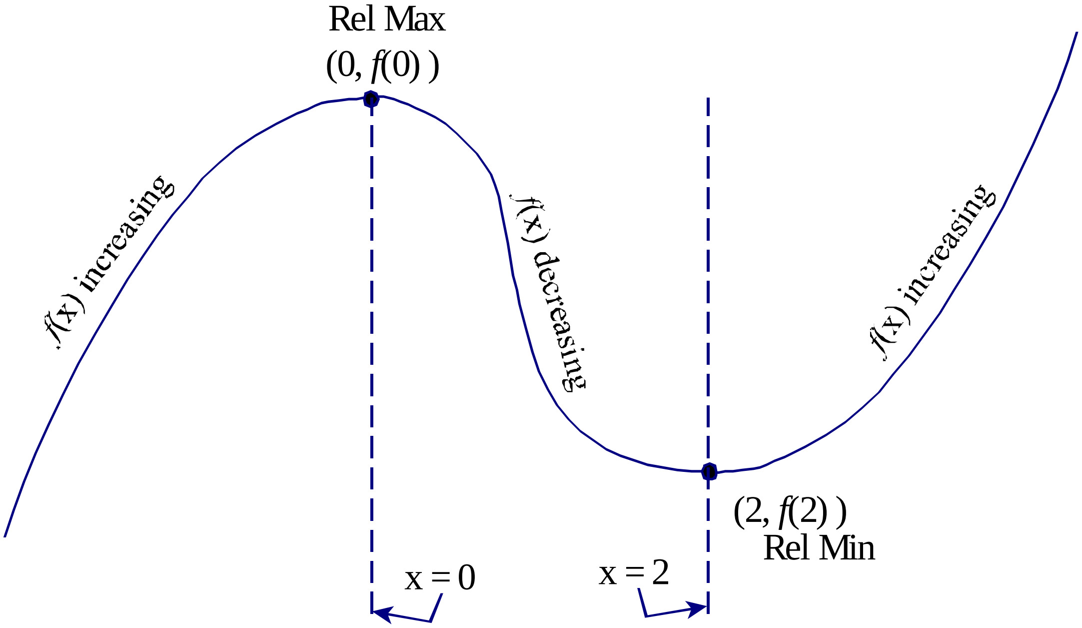
$f(x)$ is **increasing** on the intervals $(-\infty, 0)$ and $(2, \infty)$

(Because $f'(x)$ is **positive** on these intervals)

$f(x)$ is **decreasing** on the interval $(0, 2)$

(Because $f'(x)$ is **negative** on this interval)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



$f(x)$ is **increasing** on the intervals $(-\infty, 0)$ and $(2, \infty)$

(Because $f'(x)$ is **positive)**

$f(x)$ is **decreasing** on the interval $(0, 2)$

(Because $f'(x)$ is **negative)**

$(0, f(0)) = (0, 6)$ **Relative Max**

$(2, f(2)) = (2, 2)$ **Relative Min**

2. $f(x) = \frac{3}{11}x^{\frac{11}{3}} - \frac{48}{5}x^{\frac{5}{3}} + 2x + 3.$ [1]Identify the intervals on which $f(x)$ is concave up/concave down, and identify all points of inflection.

i. Compute $f''(x)$ and find possible points of inflection.

$f'(x) = x^{\frac{8}{3}} - 16x^{\frac{2}{3}}$

$f''(x) = \frac{8}{3}x^{\frac{5}{3}} - \frac{32}{3}x^{-\frac{1}{3}} = \frac{8x^{\frac{5}{3}}}{3} - \frac{32}{3x^{\frac{1}{3}}} = \frac{8x^{\frac{5}{3}}}{3}\frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}} - \frac{32}{3x^{\frac{1}{3}}} = \frac{8x^2-32}{3x^{\frac{1}{3}}}$

i.e., $f''(x) = \frac{8x^2-32}{3x^{\frac{1}{3}}}$

Find possible points of inflection.

a. "Type a" $(f''(c) = 0)$

Set $f''(x) = \frac{8x^2-32}{3x^{\frac{1}{3}}} = 0$

$\Rightarrow 8x^2 - 32 = 0$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x+2)(x-2)$

$\Rightarrow x = -2,\ x = 2$ possible points of inflection.

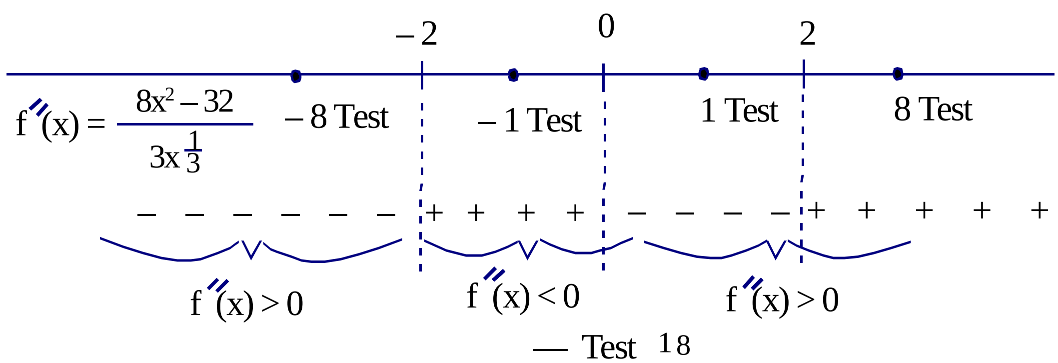
b. "Type b" $(f''(c)$ undefined)

Set denominator $3x^{\frac{1}{3}} = 0$

$\Rightarrow x = 0$ possible point of inflection.

ii. Draw a sign graph of $f''(x)$, using the possible points of inflection. to partition the x-axis

iii. From each interval select a "test point" to plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-2, 0)$ and $(2, \infty)$

(Because $f''(x)$ is **positive** on these intervals)

$f(x)$ is **concave down** on the intervals $(-\infty, -2)$ and $(0, 2)$

(Because $f''(x)$ is **negative** on this interval.

Because $f(x)$ DOES change concavity at the points $x = -2$, $x = 0$, and $x = 2$, **the points** $(-2, f(-2))\,;(0, f(0))\,;$ **and** $(2, f(2))$ **ARE points of inflection.**

3. $f(x) = 4x^3 + 9x^2 - 12x + 3$ on the interval $[-3, 0]$. Find the absolute maximum value and absolute minimum value of $f(x)$.

Since $f(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[-3, 0]$, we can use the Absolute Max/Min Value Test

i. Compute $f'(x)$ and find the critical numbers

$f'(x) = 12x^2 + 18x - 12$

a. "Type a" ($f'(c) = 0$)

Set $f'(x) = 12x^2 + 18x - 12 = 0$

$\Rightarrow 12x^2 + 18x - 12 = 0$

$\Rightarrow 2x^2 + 3x - 2 = 0$

$\Rightarrow (2x - 1)(x + 2) = 0$

$\Rightarrow x = -2;\ x = \frac{1}{2}$ "type a" crit. numbers

Alternatively: To find the critical numbers, we take the equation $2x^2 + 3x - 2 = 0$ and plug the coefficients into the quadratic equation:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-2)}}{2(2)} = \frac{3 \pm \sqrt{25}}{4}$

$\Rightarrow x = -2;\ x = \frac{1}{2}$ "type a" crit. numbers

Since $x = \frac{1}{2}$ is not in the interval $[-3, 0]$, we discard $x = \frac{1}{2}$ as a critical number.

b. "Type b" ($f'(c)$ is undefined)

No "type b" critical numbers.

ii. Plug critical numbers and endpoints into the original function.

$f(-3) = 4(-3)^3 + 9(-3)^2 - 12(-3) + 3 = 12$

$f(-2) = 4(-2)^3 + 9(-2)^2 - 12(-2) + 3 = 31 \leftarrow$ Abs Max Value

$f(0) = 4(0)^3 + 9(0)^2 - 12(0) + 3 = 3 \leftarrow$ Abs Min Value

Abs Max Value $= 31$
(attained at $x = -2$)

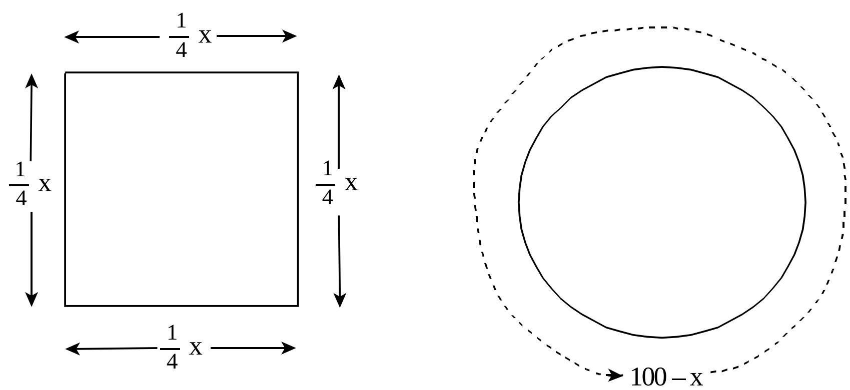
Abs Min Value $= 3$
(attained at $x = 0$)

4. Farmer Joe has 100 yards of fencing. He will use some of the fencing to construct a square pen and he will use the rest of the fencing to construct a circular pen. How much fencing should he use to construct each if he wants the total enclosed area to be as large as possible?

   Let x be the amount of fencing that he will use on the square pen

   Then $(100 - x)$ is the amount of fencing that he will use on the circular pen

   Then the circumference of the circular pen will be $C = 2\pi r = 100 - x$



   Let A be the total area enclosed.

   Then $A =$ (area of the square) + (area of the circle) $= \left(\frac{x}{4}\right)^2 + \pi r^2$

   $A = \left(\frac{x}{4}\right)^2 + \pi r^2$

   We need to know what r is in terms of x

   Recall that the circumference is $C = 2\pi r = 100 - x$

   $\Rightarrow 2\pi r = 100 - x$

   $\Rightarrow r = \frac{100-x}{2\pi}$

   Thus, Area is given by:

   $A(x) = \left(\frac{x}{4}\right)^2 + \pi r^2 = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{100-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{1}{4\pi}(100-x)^2$
   i.e., $A(x) = \frac{x^2}{16} + \frac{1}{4\pi}(100-x)^2$

   Again, we want to find out: How much fencing should be used to construct each pen, if the total enclosed area to be as large as possible?

   (Use $\pi = \frac{22}{7}$ in your final calculations)

   3. Determine the restrictions on x

      0 yards $\leq x \leq$ 100 yards

   4. Maximize the area, using the techniques of calculus.

      Note that since $A(x)$ is [1]continuous on the [2]closed, [3]finite interval [0 yards, 100 yards], we can use the Absolute Max/Min Value Test.

i) Compute $A'(x)$ and find the critical numbers

$A'(x) = \frac{1}{8}x + \frac{1}{2\pi}(100-x)(-1) = \frac{1}{8}x + \frac{1}{2\pi}(x-100) = \left(\frac{1}{8}+\frac{1}{2\pi}\right)x - \frac{100}{2\pi} = \frac{\pi+4}{8\pi}x - \frac{100}{2\pi}$

i.e., $A'(x) = \frac{\pi+4}{8\pi}x - \frac{100}{2\pi}$

Find critical numbers

a) "Type a" ($A'(x) = 0$)

$A'(x) = \frac{\pi+4}{8\pi}x - \frac{100}{2\pi} = 0$

$\Rightarrow \frac{\pi+4}{8\pi}x = \frac{100}{2\pi}$

$\Rightarrow \frac{\pi+4}{4}x = 100$

$\Rightarrow x = \frac{400}{\pi+4} = \frac{400}{\left(\frac{22}{7}\right)+4} = \frac{400}{\frac{22}{7}+\frac{28}{7}} = \frac{400}{\left(\frac{50}{7}\right)} = \frac{7}{50}\frac{400}{1} = 56$

i.e., $x = 56$ critical number

b) "Type b" ($A'(x)$ undefined)

(None)

ii) Plug critical numbers and endpoints into the **original** function

$A(0 \text{ yds}) = \frac{(0 \text{ yds})^2}{16} + \frac{1}{4\pi}(100 \text{ yds} - (0 \text{ yds}))^2 = \frac{2500}{\pi} \text{ yds}^2 \approx \frac{2500}{\left(\frac{22}{7}\right)} \text{ yds}^2 = \frac{7}{22}\frac{2500}{1} \text{ yds}^2$

$= \frac{8750}{11} \text{ yds}^2 = \left(795 + \frac{5}{11}\right) \text{ yds}^2 \longleftarrow$ Absolute Maximum Area

$A(56 \text{ yds}) = \frac{(56)^2}{16} + \frac{1}{4\pi}(100-(56))^2 = 196 + \frac{1}{4\pi}(44)^2 \approx 196 + \frac{1}{4\left(\frac{22}{7}\right)}(44)^2 = 196 + \frac{7}{88}(44)^2$

$= 196 + (7)(22) = 350$

$A(100 \text{ yds}) = \frac{(100)^2}{16} + \frac{1}{4\pi}(100-(100))^2 = 625 \text{ yds}$

5. Make sure that we've answered the original question

"How much fencing should he use to construct each . . . "
Since $x = 0$ yds yields the maximum total area, and since x is the amount of fencing that he will use on the square pen, **He should use 0 yards of fencing on the square pen and 100 yards of fencing on the circular pen.**