MTH 1126 - Test #2 - Solutions

Spring 2017

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute the arclength of the graph of the function $f(x) = \frac{8}{3}x^{\frac{3}{2}} + 4$ from the point $(0,4)$ to the point $(3, f(3))$.

   Use the formula: Arc Length $= \int_a^b \sqrt{1 + (f'(x))^2}dx$

   $f'(x) = 4x^{\frac{1}{2}}$

   $(f'(x))^2 = \left(4x^{\frac{1}{2}}\right)^2 = 16x$

   $\Rightarrow$Arc Length $= \int_a^b \sqrt{1 + (f'(x))^2}dx = \int_{x=0}^{x=3} \sqrt{1+16x}dx = \int_{x=0}^{x=3} \underbrace{(1+16x)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{dx}_{\frac{1}{16}du}$

   $$\begin{array}{|rrcl|} u & = & 1+16x \\ \Rightarrow & du & = & 16 \\ \Rightarrow & \frac{1}{16}du & = & dx \\ & & & \text{When } x=0,\ u = 1+16(0) = 1 \\ & & & \text{When } x=3,\ u = 1+16(3) = 49 \end{array}$$

   $= \int_{u=1}^{u=49} u^{\frac{1}{2}} \frac{1}{16}du = \frac{1}{16}\int_{u=1}^{u=49} u^{\frac{1}{2}}du = \frac{1}{16}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=1}^{u=49} = \frac{1}{24}(49)^{\frac{3}{2}} - \frac{1}{24}(1)^{\frac{3}{2}} = \frac{343}{24} - \frac{1}{24}$

   $= \frac{342}{24} = \frac{57}{4}$

   $$\boxed{\text{i.e., Arclength} = \frac{342}{24} = \frac{57}{4}}$$

2. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = 1 - x^2$ and $g(x) = -x + 1$.

First, graph the functions and find the points of intersection.

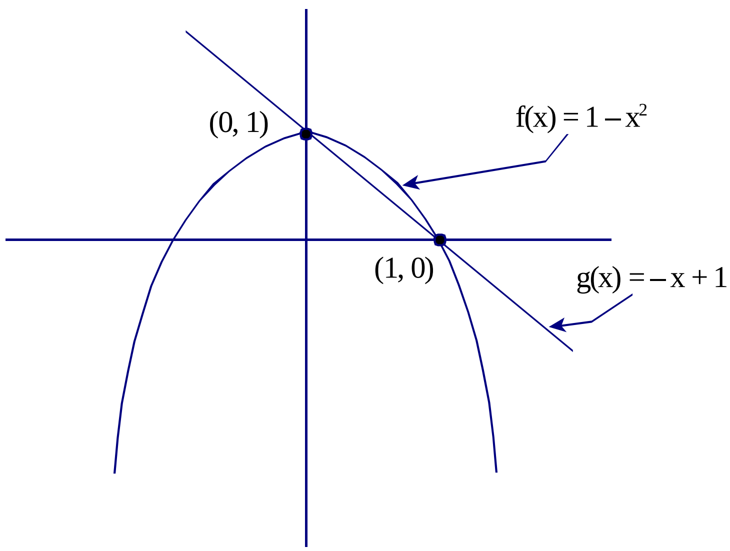
$y = 1 - x^2 = -x + 1$

$\Rightarrow -x^2 + x = 0$

$\Rightarrow x(1 - x) = 0$

$x = 0$; $x = 1$

Points of intersection are $(0, 1)$ and $(1, 0)$.



The bounded region spans the interval $[0, 1]$ on the x-axis. Over this interval, $f(x) = 1 - x^2$ is greater than $g(x) = -x + 1$. Hence the area is given by:

$\int_0^1 [(1 - x^2) - (-x + 1)]\, dx = \int_0^1 (-x^2 + x)\, dx = \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^1$

$= \left(-\frac{1}{3}(1)^3 + \frac{1}{2}(1)^2\right) - \left(-\frac{1}{3}(0)^3 + \frac{1}{2}(0)^2\right) = \frac{1}{6}$

i.e., bounded area $= \frac{1}{6}$

3. Find the area bounded by the graphs of $f(x) = 4x - x^2$ and $g(x) = x$. (Partition the appropriate interval, sketch the i^{th} rectangle, build the Riemann Sum, derive the appropriate integral.)

Graph the functions and find the points of intersection.

To graph $f(x) = 4x - x^2$, note that [1]it is a parabola and [2]its maximum will be at the critical number.

Observe: $f'(x) = 4 - 2x$

Setting $f'(x) = 0$ (to find the critical number), we have: $4 - 2x = 0$

$\Rightarrow 4 = 2x \Rightarrow x = 2$ is the critical number.

Hence, the maximum, or vertex, will be $(2, f(2)) = (2, 4)$

Notice also, that to get the x-intercepts of $f(x)$, we can set $f(x) = 0$, which yields:

$4x - x^2 = 0 \Rightarrow x = 0$ and $x = 4$ are the x-intercepts.

To find the points of intersection of the graphs $f(x)$ and $g(x)$, we set $f(x) = g(x)$, and this will give us the y-coordinates of the points of intersection.

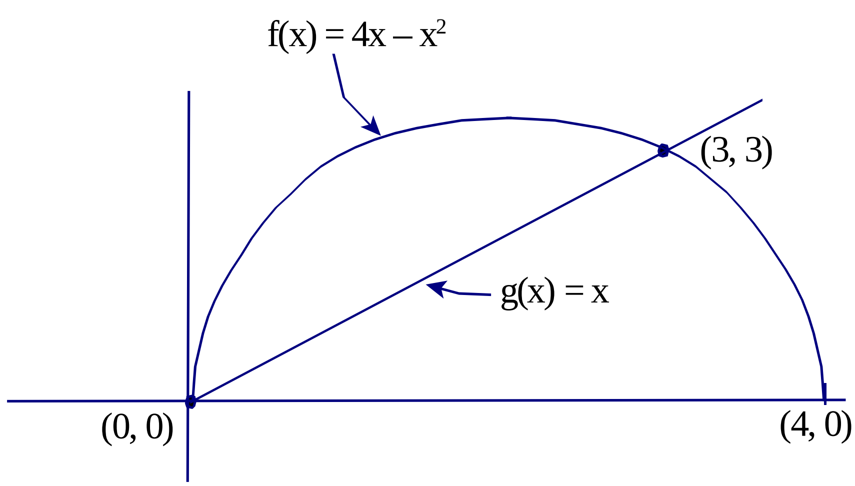
$y = 4x - x^2 = x$

$\Rightarrow 3x - x^2 = 0$

$\Rightarrow x(3 - x) = 0.$

$\Rightarrow x = 0$; and $x = 3$.

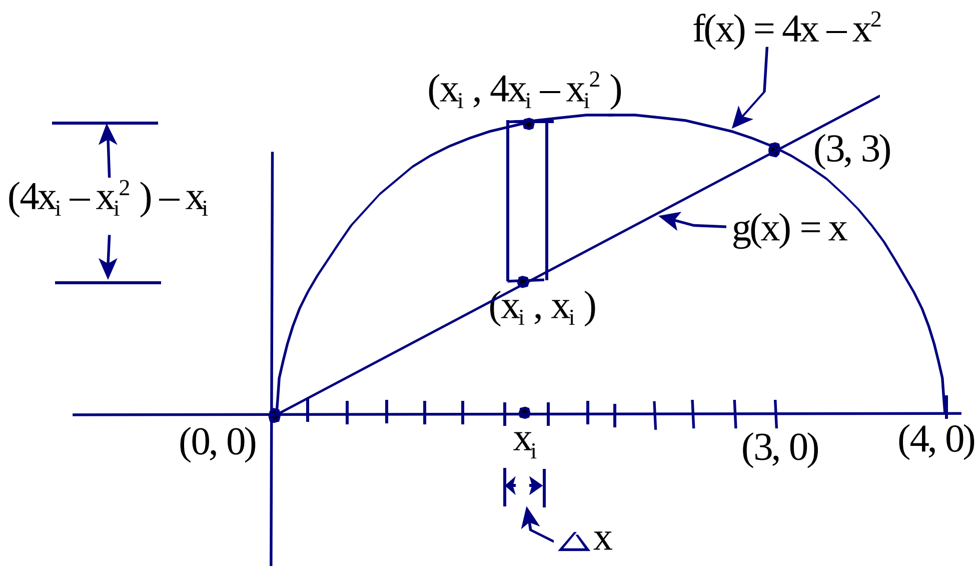
Points of intersection: $(0, 0)$ and $(3, 3)$.



The rectangles span the interval $[0, 3]$ on the x-axis, so we will partition that interval into sub-intervals of length Δx.

The area of the i^{th} rectangle is $\underbrace{\left(\left(4x_i - x_i^2\right) - x_i\right)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}} = \left(3x_i - x_i^2\right)\Delta x$

(see below)



To approximate the area of the bounded region, we add the areas of the rectangles:

$A \approx \sum_{i=1}^{n} \left(3x_i - x_i^2\right)\Delta x$

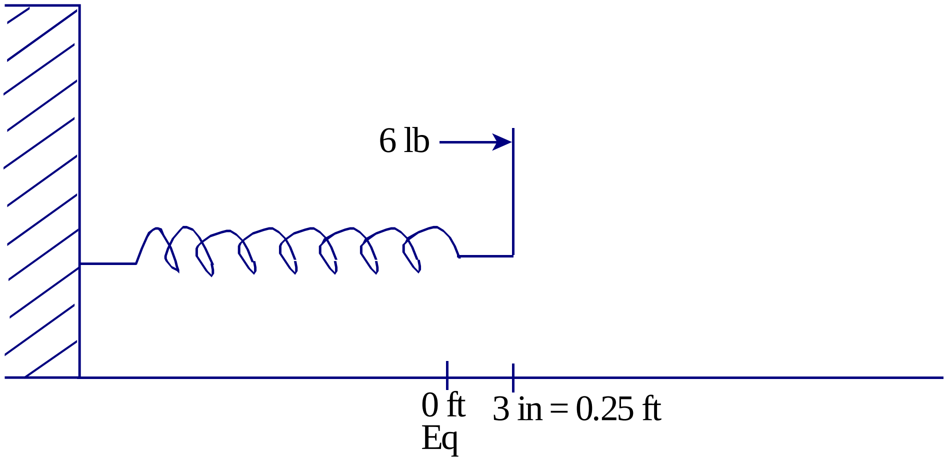
To get the exact area, we let $\Delta x \to 0$.

$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \left(3x_i - x_i^2\right)\Delta x = \int_0^3 \left(3x - x^2\right) dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$

$= \left(\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3\right) - \left(\frac{3}{2}(0)^2 - \frac{1}{3}(0)^3\right) = \frac{9}{2}$

i.e., bounded area $= \frac{9}{2}$

4. Six pounds of force is required to stretch a spring 3 inches past the point of equilibrium. How much work is done stretching the free end of the spring from 3 inches past equilibrium to 12 inches past the point of equilibrium? (Partition the appropriate interval, compute F_i, build the Riemann Sum, derive the appropriate integral.)



First, we have to find the spring constant k, using the values $F = 6$ lb and

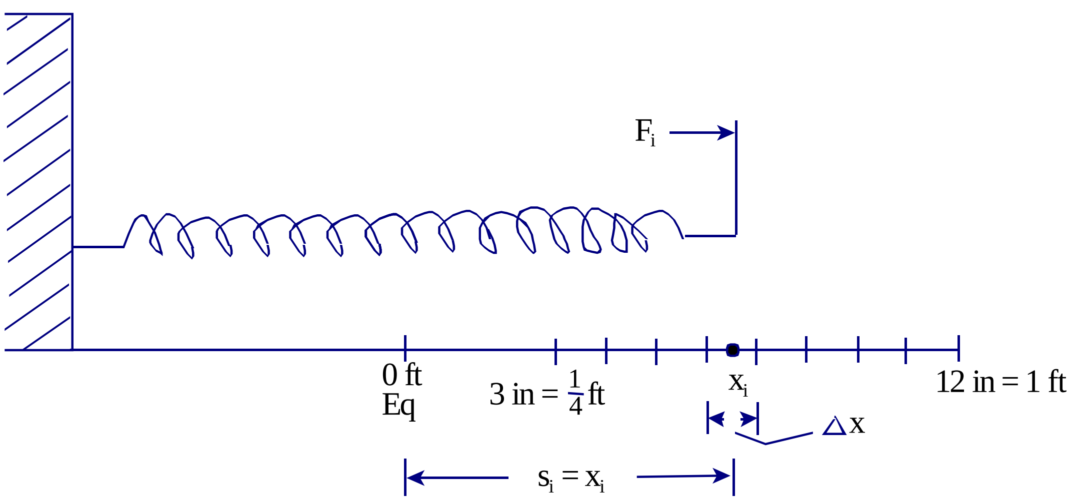
$s = 3$ inches $= \frac{1}{4}$ ft $= 0.25$ ft

From Hooke's Law $(F = ks)$ we have $k = \frac{F}{s} = \frac{6 \text{ lb}}{0.25 \text{ ft}} = 24 \ \frac{\text{lb}}{\text{ft}}$

i.e., $k = 24 \ \frac{\text{lb}}{\text{ft}}$

Hence, we have: $F = 24\frac{\text{lb}}{\text{ft}}s$

Next, partition the interval, over which the work is to be performed, and compute W_i, the work done stretching the spring from one side of the i^{th} sub-interval to the other side of the i^{th} sub-interval. (see below)



$W_i = F_i d_i$

Here, d_i is the distance over which the work W_i is performed

$d_i = \Delta x$

$F_i = ks_i = 24\frac{\text{lb}}{\text{ft}} x_i$

Hence, $W_i = F_i d_i = 24\frac{\text{lb}}{\text{ft}} x_i \Delta x$

i.e., $W_i = 24\frac{\text{lb}}{\text{ft}} x_i \Delta x$

The total work, W_T, is approximately the sum of the work done stretching the spring across each sub-interval.

$W_T \approx \sum_{i=1}^{n} 24\frac{\text{lb}}{\text{ft}} x_i \Delta x$

$W_T = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 24\frac{\text{lb}}{\text{ft}} x_i \Delta x = \int_{\frac{1}{4}\text{ ft}}^{1\text{ ft}} 24\frac{\text{lb}}{\text{ft}} x\, dx = 24\frac{\text{lb}}{\text{ft}} \int_{\frac{1}{4}\text{ ft}}^{1\text{ ft}} x\, dx = 24\frac{\text{lb}}{\text{ft}} \left[\frac{x^2}{2}\right]_{\frac{1}{4}\text{ ft}}^{1\text{ ft}}$

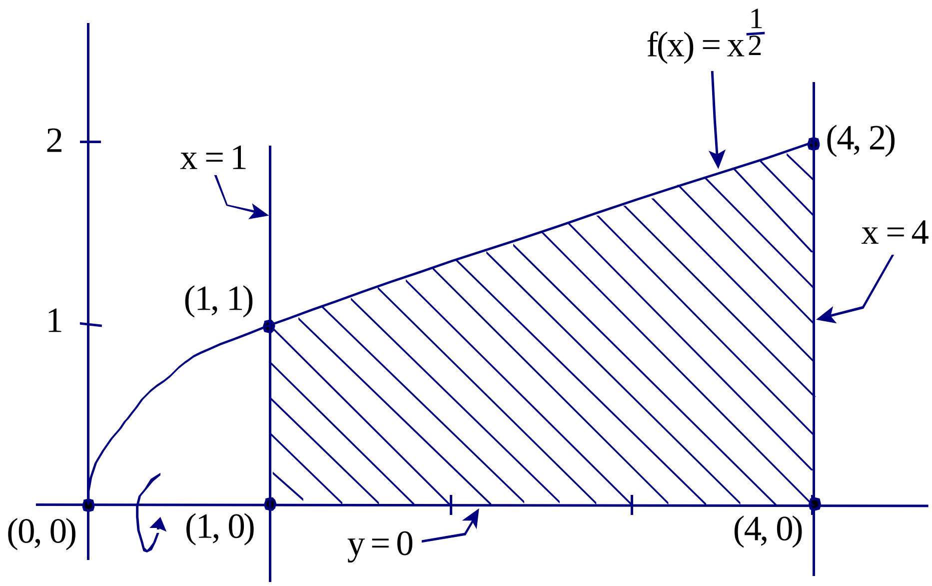
$= 24\frac{\text{lb}}{\text{ft}} \left[\left(\frac{(1\text{ ft})^2}{2}\right) - \left(\frac{\left(\frac{1}{4}\text{ ft}\right)^2}{2}\right)\right] = 24\frac{\text{lb}}{\text{ft}} \left(\frac{15}{32}\text{ ft}\right) = \frac{45}{4}\text{ lb ft}$

i.e., $W_T = \frac{45}{4}$ lb ft

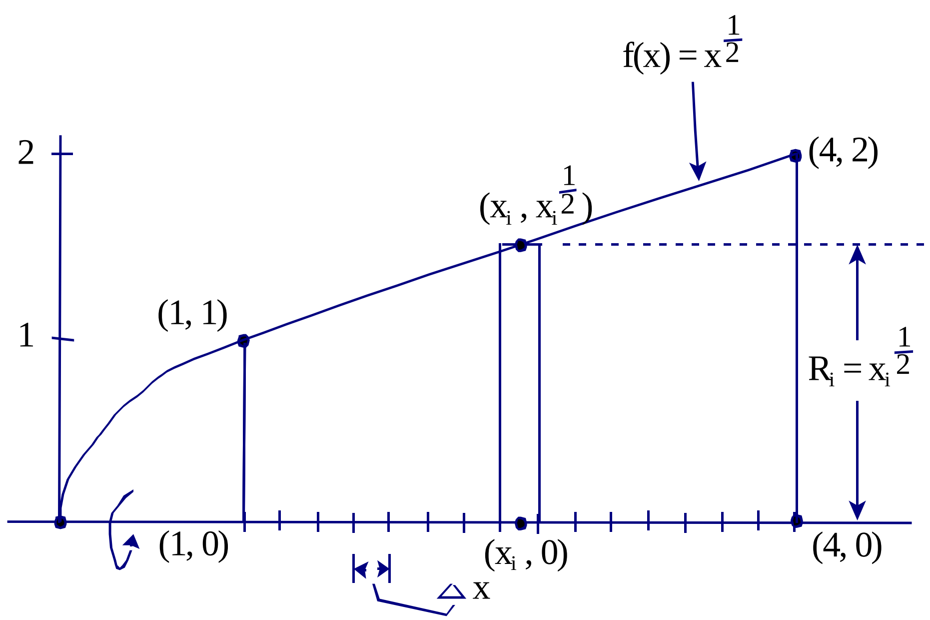
5. Use the "disc method" to compute the volume of the solid of revolution generated by revolving the region bounded by the graphs of $f(x) = x^{\frac{1}{2}}$, $x = 1$, $x = 4$, and the x-axis, about the x-axis. (For your information: the equation of the x-axis is $y = 0$.)

   Use the "five step method" (partition the interval, sketch the i^{th} rectangle, form the sum, take the limit)

   i. First, graph the bounded area.



   ii. Sketch a rectangle perpendicular (perpen-"disc"-ular) to the axis of revolution and partition the interval spanned by the rectangles.



iii. Revolve the i$^{\text{th}}$ rectangle about the axis of revolution.

Vol. of i$^{\text{th}}$ disc $= \pi R_i^2 \Delta x = \pi \left(x_i^{\frac{1}{2}} \right)^2 \Delta x = \pi \left(x_i \right) \Delta x$

iv. Approximate the volume of the solid of revolution by adding up the volumes of the discs

$Vol \approx \sum_{i=1}^{n} \pi x_i \Delta x$

v. Let $\Delta x \to 0$

$Vol \approx \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi x_i \Delta x = \int_{x=1}^{x=4} \pi x dx$

$$= \pi \left[\frac{x^2}{2} \right]_{x=1}^{x=4} = \pi \frac{(4)^2}{2} - \pi \frac{(1)^2}{2} = \frac{15\pi}{2}$$

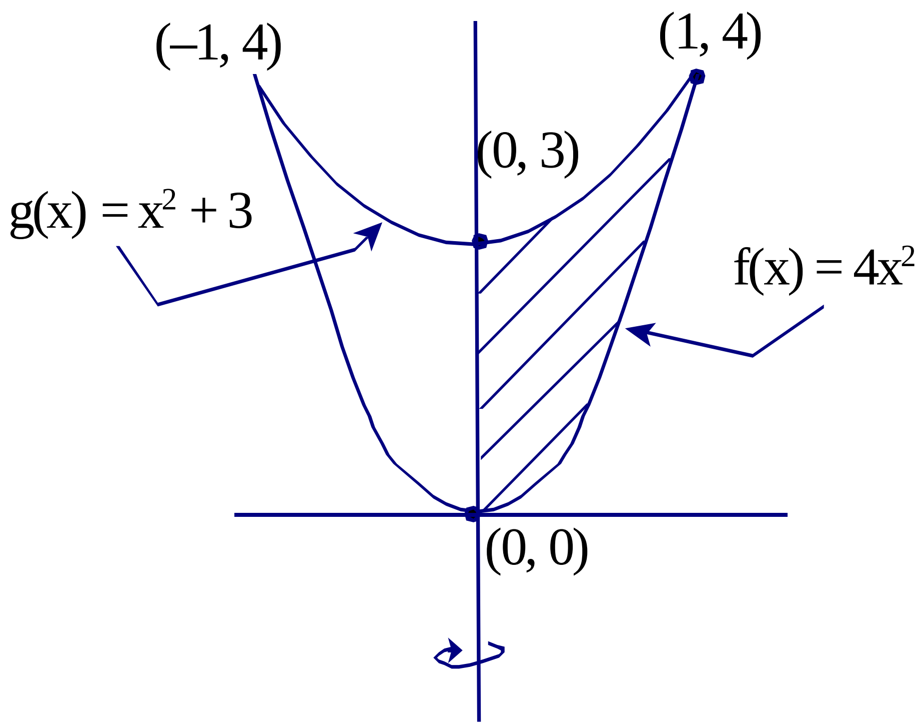
i.e., Volume $= \frac{15\pi}{2}$

6. Use the "shell method" to compute the volume of the solid of revolution generated by revolving the region described below about the y-axis.

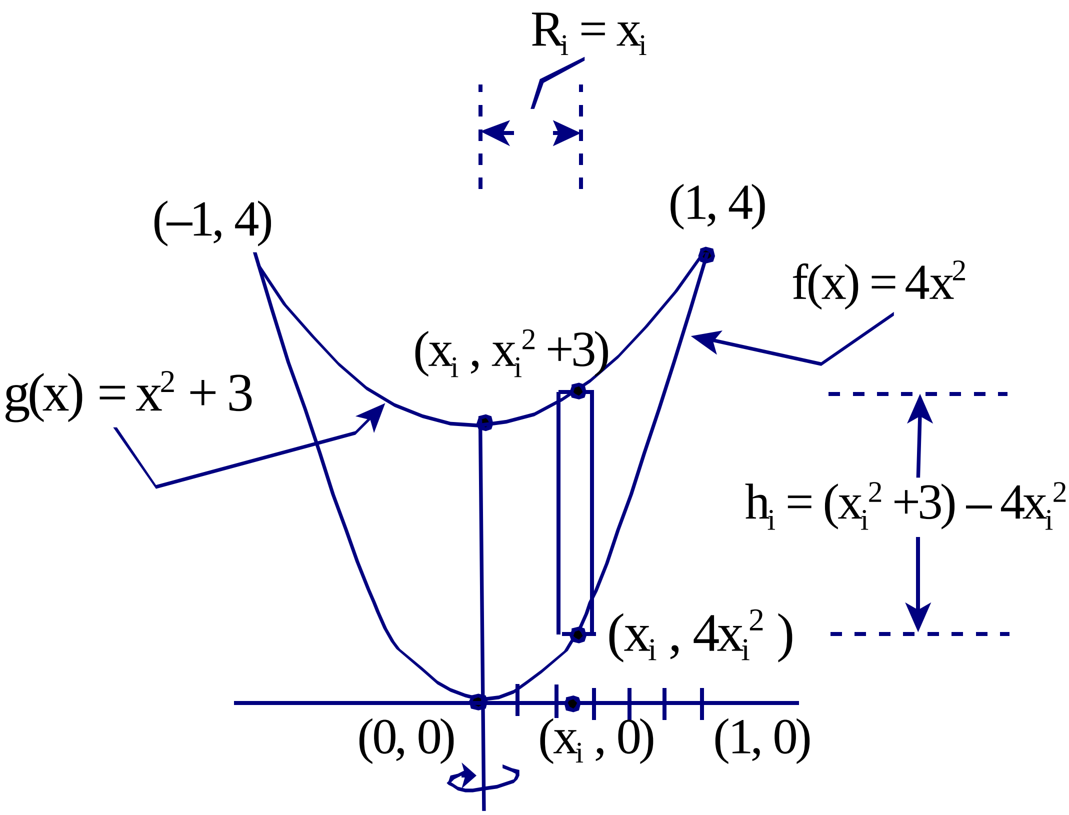
   The region lies to the right of the y-axis and is bounded by the graph $f(x) = x^2 + 3$, the y-axis, and the graph $g(x) = 4x^2$.

   Use the "five step method" (partition the interval, sketch the i^{th} rectangle, form the sum, take the limit)

   i. First, graph the bounded area.
   To find the points of intersection, set the y-coordinates equal to one another.
   $y = x^2 + 3 = 4x^2$
   $\Rightarrow -3x^2 + 3 = 0$
   $\Rightarrow x^2 - 1 = 0$
   $\Rightarrow (x+1)(x-1) = 0$
   $\Rightarrow x = -1;\ x = 1$



   ii. Sketch a rectangle *parallel* to the axis of revolution ("shell - parallel"), and partition the interval spanned by the rectangles



iii. Revolve the i$^{\text{th}}$ rectangle about the axis of revolution to form the i$^{\text{th}}$ shell.

Vol. i$^{\text{th}}$ shell $= 2\pi R_i h_i \Delta x = 2\pi x_i \left((x_i^2 + 3) - 4x_i^2 \right) \Delta x$

$$= 2\pi x_i \left(3 - 3x_i^2\right) \Delta x = 2\pi \left(3x_i - 3x_i^3\right) \Delta x$$

iv. Approximate the volume of the solid of revolution by adding the volumes of the shells.

$Vol \approx \sum_{i=1}^{n} 2\pi \left(3x_i - 3x_i^3\right) \Delta x$

v. Let $\Delta x \to 0$

$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 2\pi \left(3x_i - 3x_i^3\right) \Delta x = \int_{x=0}^{x=1} 2\pi \left(3x - 3x^3\right) dx$

$= 2\pi \left[\frac{3}{2}x^2 - \frac{3}{4}x^4\right]_{x=0}^{x=1}$

$= 2\pi \left(\frac{3}{2}(1)^2 - \frac{3}{4}(1)^4\right) - 2\pi \left(\frac{3}{2}(0)^2 - \frac{3}{4}(0)^4\right)$

$= \frac{3\pi}{2}$

i.e., $Vol = \frac{3\pi}{2}$