Formulas You Should Memorize (and I do mean Memorize!)

Summer 1997

Pat Rossi Name ____________________

Formulas From Calculus

1. $\frac{d}{dx}[\sin(x)] = \cos(x)$
2. $\frac{d}{dx}[\cos(x)] = -\sin(x)$
3. $\frac{d}{dx}[\tan(x)] = \sec^2(x)$
4. $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$
5. $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
6. $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

   Remark 1 *Note that if you know the derivatives of* $\sin(x)$, $\tan(x)$, *and* $\sec(x)$, *then the derivatives of the corresponding "co-functions"* $\cos(x)$, $\cot(x)$, *and* $\csc(x)$ *are found by:*

   (a) Changing the sign, and

   (b) Replacing each factor of the derivative with its co-function.

   Example 1 $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x) \implies \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

7. $\int \sin(x)\ dx = -\cos(x) + C$
8. $\int \cos(x)\ dx = \sin(x) + C$
9. $\int \tan(x)\ dx = \ln|\sec(x)| + C = -\ln|\cos(x)| + C$
10. $\int \cot(x)\ dx = \ln|\sin(x)| + C$
11. $\int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C$
12. $\int \csc(x)\ dx = \ln|\csc(x) - \cot(x)| + C$
13. $\int \sec^2(x)\ dx = \tan(x) + C$
14. $\int \csc^2(x)\ dx = -\cot(x) + C$
15. $\int \sec(x)\tan(x)\ dx = \sec(x) + C$
16. $\int \csc(x)\cot(x)\ dx = -\csc(x) + C$
17. $\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$
18. $\frac{d}{dx}[\ln(u)] = \frac{1}{u} \cdot \frac{du}{dx}$

19. $\int e^{u}\,du = e^{u} \ + C$

20. $\int \frac{1}{u}\,du = \ln|u| \ + C$

21. $\int u^{-1}\,du = \ln|u| \ + C$

22. $\int \ln(u)\ du = u\ln(u) - u + C$

23. $\frac{d}{dx}\left[\sin^{-1}(u)\right] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

24. $\frac{d}{dx}\left[\cos^{-1}(u)\right] = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

25. $\frac{d}{dx}\left[\tan^{-1}(u)\right] = \frac{1}{1+u^2} \cdot \frac{du}{dx}$

26. $\frac{d}{dx}\left[\cot^{-1}(u)\right] = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$

27. $\frac{d}{dx}\left[\sec^{-1}(u)\right] = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

28. $\frac{d}{dx}\left[\csc^{-1}(u)\right] = -\frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

Remark 2 *Note that if you know the derivatives of the inverse trig functions* $\sin^{-1}(x)$, $\tan^{-1}(x)$, *and* $\sec^{-1}(x)$, *the derivatives of the corresponding inverse "co-functions" can be found by changing the sign.*

29. $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) \ + C$

30. $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) \ + C$

31. $\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) \ + C$

Essential Trigonometric Identities

32. $\sin^2(x) + \cos^2(x) = 1$

33. $\tan^2(x) + 1 = \sec^2(x)$

34. $\cot^2(x) + 1 = \csc^2(x)$

Remark 3 *Identities 32-34 are the so-called "Pythagorean Identities".*

35. $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{1}{\cot(x)}$

36. $\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$

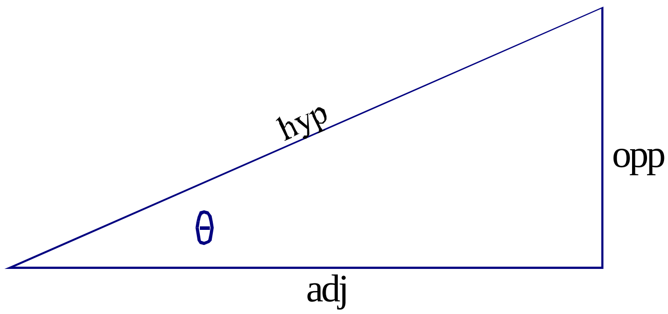
37. $\sec(x) = \frac{1}{\cos(x)}$

38. $\csc(x) = \frac{1}{\sin(x)}$

39. $\sin(2x) = 2\sin(x)\cos(x)$

40. $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$

Right Triangle Trig.



41. $\sin(\theta) = \frac{\text{opp}}{\text{hyp}}$

42. $\cos(\theta) = \frac{\text{adj}}{\text{hyp}}$

43. $\tan(\theta) = \frac{\text{opp}}{\text{adj}}$

44. $\cot(\theta) = \frac{\text{adj}}{\text{opp}}$

45. $\sec(\theta) = \frac{\text{hyp}}{\text{adj}}$

46. $\csc(\theta) = \frac{\text{hyp}}{\text{opp}}$

Logs and Exponential Functions

47. $\ln(xy) = \ln(x) + \ln(y)$
    (a) $\log_a(xy) = \log_a(x) + \log_a(y)$

48. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
    (a) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

49. $\ln(x^n) = n \cdot \ln(x)$
    (a) $\log_a(x^n) = n \cdot \log_a(x)$

50. $\ln(e^x) = x$
    (a) $\log_a(a^x) = x$

51. $e^{\ln(x)} = x$
    (a) $a^{\log_a(x)} = x$

52. $\ln(e) = 1$

   (a) $\log_a(a) = 1$

53. $\ln(1) = 0$

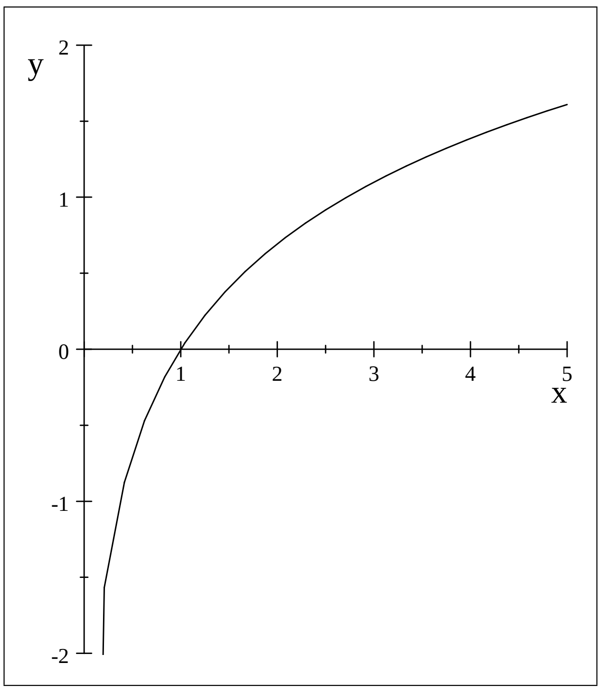
   (a) $\log_a(1) = 0$

54. $\lim_{x\to\infty} \ln(x) = \infty$

   (a) $\lim_{x\to\infty} \log_a(x) = \infty$

55. $\lim_{x\to 0^+} \ln(x) = -\infty$

   (a) $\lim_{x\to 0^+} \log_a(x) = -\infty$

$y = \ln(x)$ 

56. $e^{x+y} = e^x e^y$

   (a) $a^{x+y} = a^x a^y$

57. $\frac{e^x}{e^y} = e^{x-y}$

   (a) $\frac{a^x}{a^y} = a^{x-y}$

58. $(e^x)^n = e^{xn}$

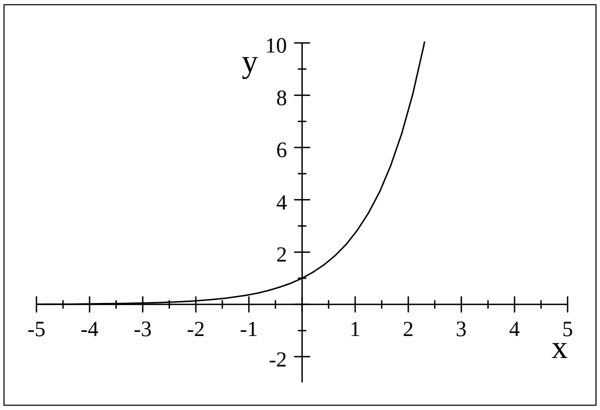
   (a) $(a^x)^n = a^{xn}$

59. $\lim_{x\to\infty} e^x = \infty$

   (a) $\lim_{x\to\infty} a^x = \infty$

60. $\lim_{x\to-\infty} e^x = 0$

   (a) $\lim_{x\to-\infty} a^x = 0$

$y = e^x$ 

61. $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

62. $\lim_{x\to 0} (1+kx)^{\frac{1}{x}} = e^k$

   (a) $\lim_{x\to 0} (1-kx)^{\frac{1}{x}} = e^{-k}$

63. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

64. $\lim_{x\to\infty} \left(1+\frac{k}{x}\right)^x = e^k$

   (a) $\lim_{x\to\infty} \left(1-\frac{k}{x}\right)^x = e^{-k}$

Remark 4 *Note that 62.a and 64.a can be obtained easily from 62 and 64 respectively, by "plugging in" $-k$ in place of k.*

65. $(xy)^n = x^n y^n$

66. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

Calculus of General Logs and Exponentials

67. $\frac{d}{dx}[a^x] = \ln(a) \cdot a^x$

68. $\frac{d}{dx}[a^u] = \ln(a) \cdot a^u \cdot \frac{du}{dx}$

69. $\int a^u \, du = \frac{1}{\ln(a)} a^u + C$

70. $\frac{d}{dx}[\log_a(x)] = \frac{1}{\ln(a)} \cdot \frac{1}{x}$

71. $\frac{d}{dx}[\log_a(u)] = \frac{1}{\ln(a)} \cdot \frac{1}{u} \cdot \frac{du}{dx}$

By-Passing U-Substitution, Given Simple Composite Functions

Suppose that $F(x)$ is an anti-derivative of $f(x)$. (i.e., suppose that $\int f(x)\,dx = F(x) + C$)

Then if k and c are constants, the following general principles hold:

$$\int f(kx)\,dx = \frac{1}{k}F(kx) + C \qquad \text{and} \qquad \int f(kx+c)\,dx = \frac{1}{k}F(kx+c) + C)$$

Some Cases in Point:

$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$

$\int \sin(kx+c)\,dx = -\frac{1}{k}\cos(kx+c) + C$

$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$

$\int \cos(kx+c)\,dx = \frac{1}{k}\sin(kx+c) + C$

$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$

$\int e^{kx+c}dx = \frac{1}{k}e^{kx+c} + C$

$\int \frac{1}{kx}dx = \frac{1}{k}\ln(kx) + C$

$\int \frac{1}{kx+c}dx = \frac{1}{k}\ln(kx+c) + C$

Series and Sums

Finite Series

72. $\sum_{i=0}^{n} r^i \cdot a = \frac{1-r^{n+1}}{1-r}$ **Finite Geometric Series** with ratio r.

73. $\sum_{i=0}^{n} (ki + a) = \frac{n+1}{2}(a_1 + a_2)$ **Arithmetic Series** with common difference k.

74. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i$

Infinite Series

75. If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. Conversely, if a_n does not go to 0, then the series must diverge.

76. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$ is the **Harmonic Series.** It diverges.

77. The series:

    (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots$ and

    (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots + \frac{(-1)^{n}}{n} + \ldots$

    are **Alternating Harmonic Series**. These series converge.

78. $\sum_{n=0}^{\infty} r^n a = a + ra + r^2 a + r^3 a + \ldots + r^n a + \ldots$ is the **Geometric Series** with ratio r.

   (a) If $|r| < 1$, the series converges, and the sum is given by $\frac{\text{first term}}{1-r}$.

   (b) If $|r| \geq 1$, the series diverges.

79. The series:

   (a) $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1} a_n + \ldots$ and

   (b) $\sum_{n=1}^{\infty} (-1)^{n} a_n = -a_1 + a_2 - a_3 + a_4 - \ldots + (-1)^{n+1} a_n + \ldots$ where:

      1. $a_n \geq 0$
      2. $a_n \geq a_{n+1}$
      3. $\lim_{n\to\infty} a_n = 0$

      are **Alternating Series.** They converge. Also:

$$\left| \sum_{n=1}^{\infty} (-1)^{n+1} a_n - \sum_{n=1}^{k} (-1)^{n+1} a_n \right| \leq |a_{k+1}|$$

   (A similar statement can be made for $\sum_{n=1}^{\infty} (-1)^{n} a_n$)

80. **Direct Comparison Test** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be such that a_n and b_n are non-negative for all but finitely many terms.

   (a) If $a_n \leq b_n$ for all but finitely many terms and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges also.

   (b) If $a_n \leq b_n$ for all but finitely many terms and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges also.

   (c) If $a_n \leq b_n$ and $\sum_{n=1}^{\infty} a_n$ converges, this tells us nothing about $\sum_{n=1}^{\infty} b_n$.

   (d) If $a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, this tells us nothing about $\sum_{n=1}^{\infty} a_n$.

81. **Limit Comparison Test** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be such that a_n and b_n are non-negative for all but finitely many terms.

   (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is **non-zero and finite** (i.e. $0 < c < \infty$), then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, or both diverge.

   (b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ or if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, we can conclude nothing. In this case, we have made a poor choice for comparison. Choose another series for comparison.

82. **p-series** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a positive constant, is the **p-series**. This series converges if $p > 1$ and diverges if $p \leq 1$.

83. **Ratio Test** Consider $\sum_{n=1}^{\infty} a_n$.

   (a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

   (c) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

84. **The Integral Test Let** $\sum_{n=1}^{\infty} a_n$ be such that a_n is non-negative for all but finitely many terms. If:

   (a) $f(x)$ is a continuous function on the interval $[1, \infty)$, with the property that $f(n) = a_n$ for $n = 1, 2, 3, \ldots$ and

   (b) $f(x)$ is decreasing on $[1, \infty)$,

   then either both $\sum_{n=1}^{\infty} a_n$ and $\int_a^b f(x)\,dx$ converge, or they both diverge.

85. The **n$^{\text{th}}$ Root Test** Consider $\sum_{n=1}^{\infty} a_n$.

   (a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then series $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then series $\sum_{n=1}^{\infty} a_n$ diverges.

   (c) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the n^{th} Root Test is inconclusive.

86. The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ is the **Taylor Series** of $f(x)$ with center x_0. Here,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Some Well-Known Taylor Series

87. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$ for all x.

88. $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{n+1} + \ldots$ for $|x| < 1$

89. $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$ for all x.

90. $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$ for all x.

91. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$ for $|x| < 1$.