MTH 2227 Test #1 - Solutions

Spring 2018

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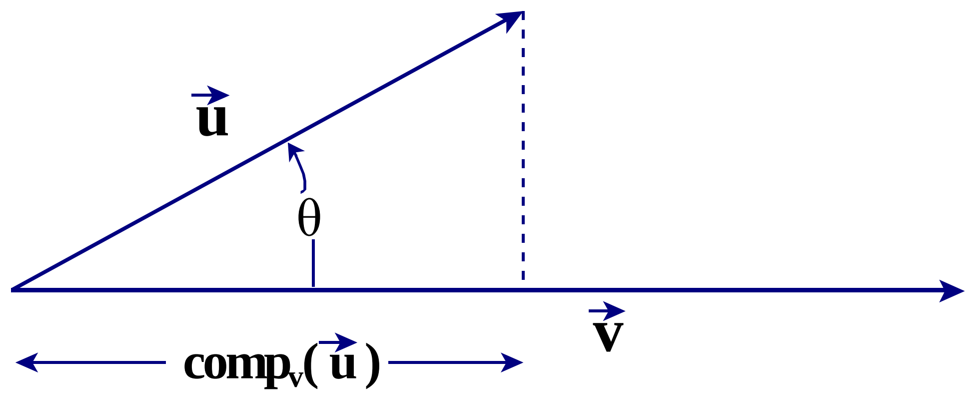
Instructions. Show clearly how you arrive at your answers. Document your work.

1. Compute $\mathbf{proj}_v(\tilde{\mathbf{u}})$ and $\mathbf{orth}_v(\tilde{\mathbf{u}})$ if $\tilde{\mathbf{u}} = \langle 2, 6\rangle$ and $\tilde{\mathbf{v}} = \langle 5, 3\rangle$.

   Recall: The component of $\tilde{\mathbf{u}}$ along $\tilde{\mathbf{v}}$, denoted $\mathbf{comp}_v(\tilde{\mathbf{u}})$, is the magnitude of $\tilde{\mathbf{u}}$ in the direction of $\tilde{\mathbf{v}}$, as shown below:



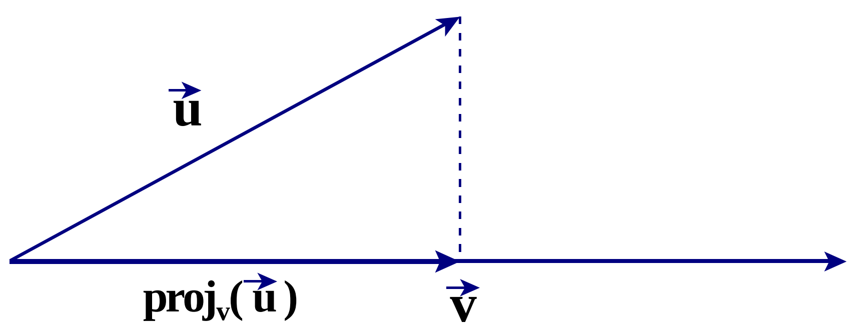
   From the picture above, we can clearly see that $\mathbf{comp}_v(\tilde{\mathbf{u}}) = |\tilde{\mathbf{u}}|\cos(\theta)$, where θ is the angle between the vectors $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$.

   Recall that $\cos(\theta) = \frac{\tilde{\mathbf{u}}\circ\tilde{\mathbf{v}}}{|\tilde{\mathbf{u}}||\tilde{\mathbf{v}}|}$

   Thus, $\mathbf{comp}_v(\tilde{\mathbf{u}}) = |\tilde{\mathbf{u}}|\cos(\theta) = |\tilde{\mathbf{u}}|\frac{\tilde{\mathbf{u}}\circ\tilde{\mathbf{v}}}{|\tilde{\mathbf{u}}||\tilde{\mathbf{v}}|} = \frac{\tilde{\mathbf{u}}\circ\tilde{\mathbf{v}}}{|\tilde{\mathbf{v}}|} = \frac{\langle 2,6\rangle\circ\langle 5,3\rangle}{|\langle 5,3\rangle|} = \frac{2\cdot 5+6\cdot 3}{\sqrt{5^2+3^2}} = \frac{28}{\sqrt{34}}$

   i.e., $\mathbf{comp}_v(\tilde{\mathbf{u}}) = \frac{28}{\sqrt{34}}$

   Recall also: The projection of $\tilde{\mathbf{u}}$ onto $\tilde{\mathbf{v}}$, denoted $\mathbf{proj}_v(\tilde{\mathbf{u}})$, is the vector that has the magnitude of $\mathbf{comp}_v(\tilde{\mathbf{u}})$ in the direction of $\tilde{\mathbf{v}}$, as shown below.

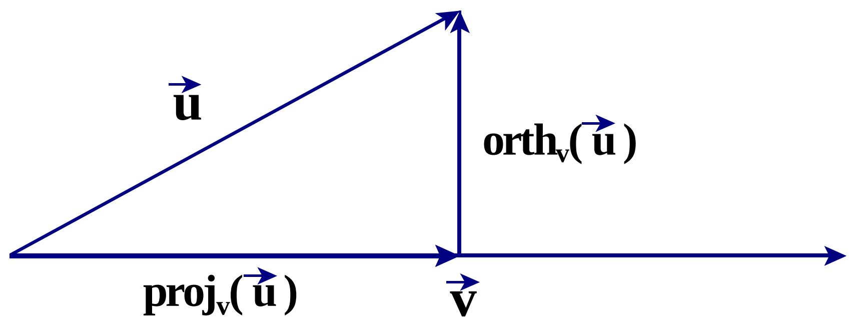


   To get $\mathbf{proj}_v(\tilde{\mathbf{u}})$, we multiply the unit vector in the direction of $\tilde{\mathbf{v}}$ by the scalar: $\mathbf{comp}_v(\tilde{\mathbf{u}})$.

   i.e., $\mathbf{proj}_v(\tilde{\mathbf{u}}) = \mathbf{comp}_v(\tilde{\mathbf{u}})\cdot\frac{\tilde{\mathbf{v}}}{|\tilde{\mathbf{v}}|} = \frac{28}{\sqrt{34}}\frac{\langle 5,3\rangle}{|\langle 5,3\rangle|} = \frac{28}{\sqrt{34}}\frac{\langle 5,3\rangle}{\sqrt{5^2+3^2}} = \frac{28}{\sqrt{34}}\frac{\langle 5,3\rangle}{\sqrt{34}} = \frac{28}{34}\langle 5,3\rangle = \frac{14}{17}\langle 5,3\rangle = \left\langle \frac{70}{17}, \frac{42}{17}\right\rangle$

i.e., $\mathbf{proj}_v\left(\tilde{\mathbf{u}}\right) = \left\langle \frac{70}{17}, \frac{42}{17} \right\rangle$

Finally, $\mathbf{orth}_v\left(\tilde{\mathbf{u}}\right)$, the component of $\tilde{\mathbf{u}}$ orthogonal to $\tilde{\mathbf{v}}$, is shown below:



Notice that $\mathbf{proj}_v\left(\tilde{\mathbf{u}}\right) + \mathbf{orth}_v\left(\tilde{\mathbf{u}}\right) = \tilde{\mathbf{u}}$

Therefore, $\mathbf{orth}_v\left(\tilde{\mathbf{u}}\right) = \tilde{\mathbf{u}} - \mathbf{proj}_v\left(\tilde{\mathbf{u}}\right) = \langle 2, 6\rangle - \left\langle \frac{70}{17}, \frac{42}{17} \right\rangle = \left\langle \frac{34}{17}, \frac{102}{17} \right\rangle - \left\langle \frac{70}{17}, \frac{42}{17} \right\rangle = \left\langle -\frac{36}{17}, \frac{60}{17} \right\rangle$

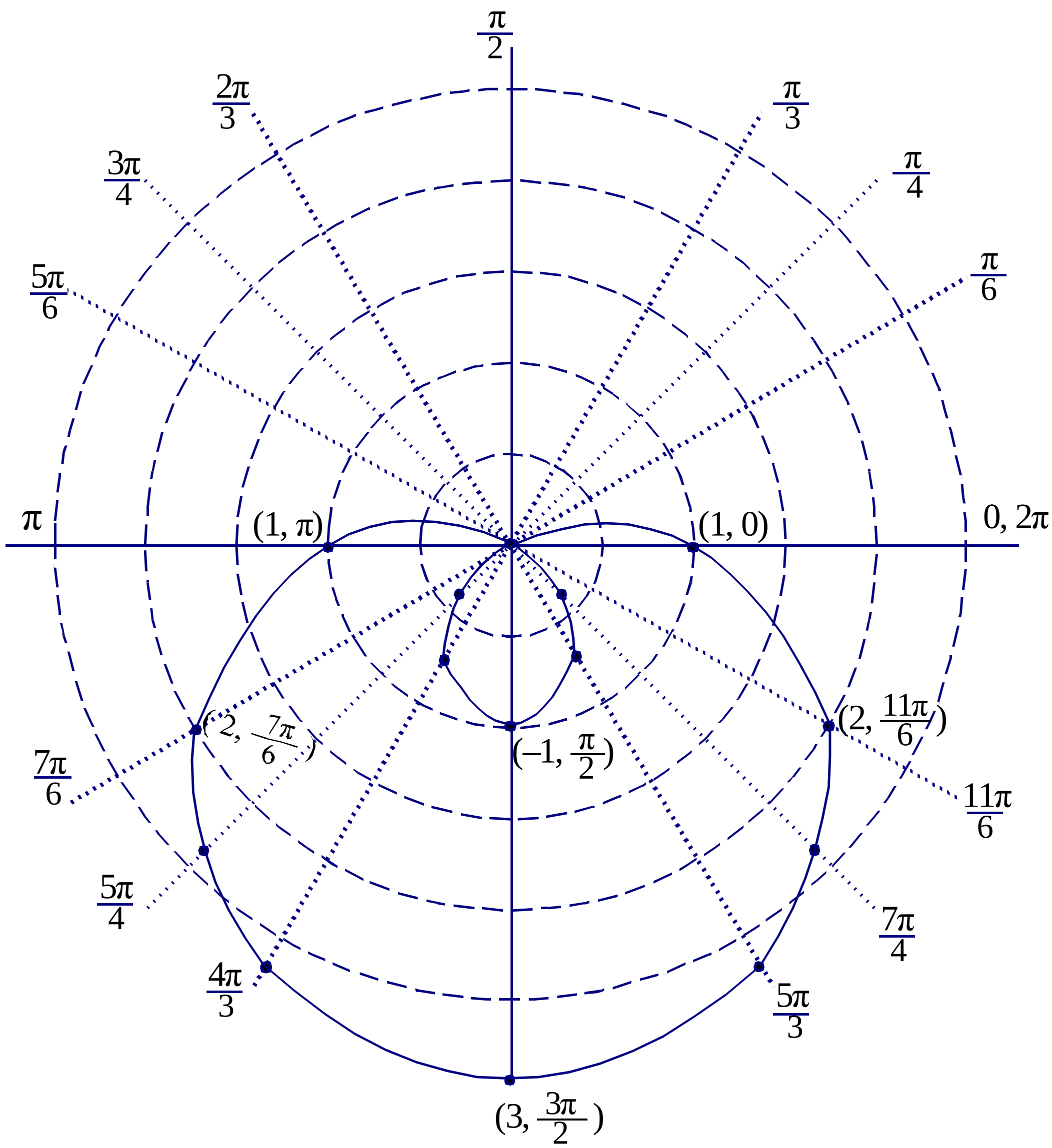
$\mathbf{proj}_v\left(\tilde{\mathbf{u}}\right) = \left\langle \frac{70}{17}, \frac{42}{17} \right\rangle$

$\mathbf{orth}_v\left(\tilde{\mathbf{u}}\right) = \left\langle -\frac{36}{17}, \frac{60}{17} \right\rangle$

2. Graph the equation $r = 1 - 2\sin(\theta)$ **Note**: $\frac{\sqrt{2}}{2} \approx 0.707$ $\frac{\sqrt{3}}{2} \approx 0.866$

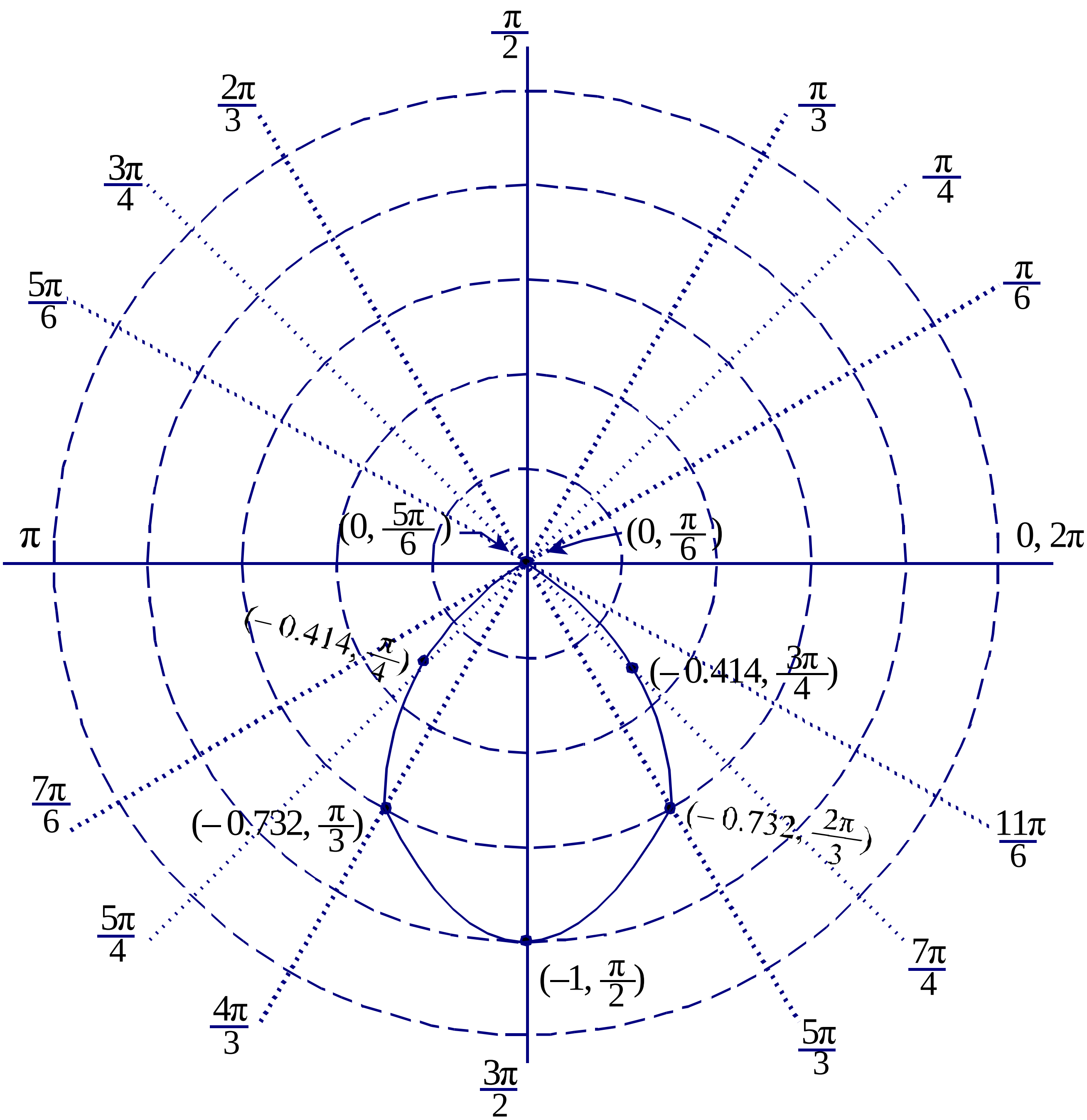
$\sqrt{2} \approx 1.414$ $\sqrt{3} \approx 1.732$

$\theta =$	$r = 1 - 2\sin(\theta)$		$\theta =$	$r = 1 - 2\sin(\theta)$
0	$1 - 2(0) = 1$			
$\frac{\pi}{6}$	$1 - 2\left(\frac{1}{2}\right) = 0$		$\frac{7\pi}{6}$	$1 - 2\left(-\frac{1}{2}\right) = 2$
$\frac{\pi}{4}$	$1 - 2\left(\frac{\sqrt{2}}{2}\right) \approx -0.414$		$\frac{5\pi}{4}$	$1 - 2\left(-\frac{\sqrt{2}}{2}\right) \approx 2.414$
$\frac{\pi}{3}$	$1 - 2\left(\frac{\sqrt{3}}{2}\right) \approx -0.732$		$\frac{4\pi}{3}$	$1 - 2\left(-\frac{\sqrt{3}}{2}\right) \approx 2.732$
$\frac{\pi}{2}$	$1 - 2(1) = -1$		$\frac{3\pi}{2}$	$1 - 2(-1) = 3$
$\frac{2\pi}{3}$	$1 - 2\left(\frac{\sqrt{3}}{2}\right) \approx -0.732$		$\frac{5\pi}{3}$	$1 - 2\left(-\frac{\sqrt{3}}{2}\right) \approx 2.732$
$\frac{3\pi}{4}$	$1 - 2\left(\frac{\sqrt{2}}{2}\right) \approx -0.414$		$\frac{7\pi}{4}$	$1 - 2\left(-\frac{\sqrt{2}}{2}\right) \approx 2.414$
$\frac{5\pi}{6}$	$1 - 2\left(\frac{1}{2}\right) = 0$		$\frac{11\pi}{6}$	$1 - 2\left(-\frac{1}{2}\right) = 2$
π	$1 - 2(0) = 1$		2π	$1 - 2(0) = 1$



3. Given the function $r = 1 - 2\sin(\theta)$ from the previous exercise, find the area bounded by the "inner loop" of the graph.

Redrawing the "inner loop" of the graph, we see that the inner loop is created as θ assumes the values between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. Thus, $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ will be the lower and upper limits of integration, respectively.



Using the formula: Area $= \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$, we have:

Area $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1-2\sin(\theta))^2\, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(4\sin^2(\theta) - 4\sin(\theta) + 1\right) d\theta$

We can't integrate $\sin^2(\theta)$ in its current form, so we use an identity to change its form:

$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$

Continuing, we have:

$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1-2\sin(\theta))^2\, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(4\sin^2(\theta) - 4\sin(\theta) + 1\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(4\frac{1-\cos(2\theta)}{2} - 4\sin(\theta) + 1\right) d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2 - 2\cos(2\theta) - 4\sin(\theta) + 1)\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3 - 2\cos(2\theta) - 4\sin(\theta))\, d\theta = \frac{1}{2}\left[3\theta - \sin(2\theta) + 4\cos(\theta)\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \frac{1}{2}\left[3\left(\frac{5\pi}{6}\right) - \sin\left(2\left(\frac{5\pi}{6}\right)\right) + 4\cos\left(\frac{5\pi}{6}\right)\right] - \frac{1}{2}\left[3\left(\frac{\pi}{6}\right) - \sin\left(2\left(\frac{\pi}{6}\right)\right) + 4\cos\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{5\pi}{2}\right) - \sin\left(\frac{5\pi}{3}\right) + 4\cos\left(\frac{5\pi}{6}\right)\right] - \frac{1}{2}\left[\left(\frac{\pi}{2}\right) - \sin\left(\left(\frac{\pi}{3}\right)\right) + 4\cos\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5\pi}{2} - \left(-\frac{\sqrt{3}}{2}\right) + 4\left(-\frac{\sqrt{3}}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{\pi}{2}\right) - \left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \pi - \frac{3}{2}\sqrt{3}$$

4. Determine which vectors (if any) are parallel and which (if any) are perpendicular (orthogonal). $\tilde{\mathbf{u}} = \langle 3, 0, -1 \rangle$; $\tilde{\mathbf{v}} = \langle 3, 3, 9 \rangle$; $\tilde{\mathbf{w}} = \langle 2, 2, 6 \rangle$.

   Recall: Two vectors in "standard (component) form" are parallel if one is a scalar multiple of the other.

   Observe: $\tilde{\mathbf{v}} = \langle 3, 3, 9 \rangle = \frac{3}{2} \langle 2, 2, 6 \rangle = \frac{3}{2}\tilde{\mathbf{w}}$

   (i.e., $\tilde{\mathbf{v}} = \frac{3}{2}\tilde{\mathbf{w}}$.)

Hence $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$ are parallel.

   Note: If the remaining vector $\tilde{\mathbf{u}}$ is perpendicular to either $\tilde{\mathbf{v}}$ or $\tilde{\mathbf{w}}$, then it must be perpendicular to both $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$, since $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$ are parallel.

   Recall: Two vectors in "standard (component) form" are perpendicular (orthogonal) if their "dot product" equals zero.

   Observe: $\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}} = \langle 3, 0, -1 \rangle \circ \langle 3, 3, 9 \rangle = 3 \cdot 3 + 0 \cdot 3 + (-1) \cdot 9 = 0$

   (i.e., $\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}} = 0$.) Hence, $\tilde{\mathbf{u}}$ is perpendicular to $\tilde{\mathbf{v}}$, and by our earlier observation, $\tilde{\mathbf{u}}$ is perpendicular to $\tilde{\mathbf{w}}$ also.

$\tilde{\mathbf{u}}$ is perpendicular to both $\tilde{\mathbf{v}}$, and $\tilde{\mathbf{w}}$.

5. Find the slope of the graph in the x-y plane, given parametrically by $x = e^t$ and $y = \sin(\pi t)$ when $t = 0$.

   Recall: the slope of such a graph is given by $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

   Thus, $\frac{dy}{dx} = \left.\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}\right|_{t=0} = \left.\frac{\frac{d}{dt}(\sin(\pi t))}{\frac{d}{dt}(e^t)}\right|_{t=0} = \left.\frac{\pi \cos(\pi t)}{e^t}\right|_{t=0} = \frac{\pi \cos(\pi(0))}{e^0} = \frac{\pi}{1} = \pi$

Slope $\frac{dy}{dx} = \pi$

6. Find the angle, θ, between the vectors $\tilde{\mathbf{u}} = \langle 4, 2, 3\rangle$ and $\tilde{\mathbf{v}} = \left\langle 1, -1, -\frac{1}{2}\right\rangle$

   Recall: $\cos(\theta) = \frac{\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}}{|\tilde{\mathbf{u}}||\tilde{\mathbf{v}}|}$, where θ is the angle between the vectors $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$

   Thus, $\theta = \arccos\left(\frac{\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}}{|\tilde{\mathbf{u}}||\tilde{\mathbf{v}}|}\right)$

   We need:

   $\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}} = \langle 4, 2, 3\rangle \circ \left\langle 1, -1, -\frac{1}{2}\right\rangle = 4 \cdot 1 + 2 \cdot (-1) + 3 \cdot \left(-\frac{1}{2}\right) = \frac{1}{2}$

   $|\tilde{\mathbf{u}}| = |\langle 4, 2, 3\rangle| = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{29}$

   $|\tilde{\mathbf{v}}| = \left|\left\langle 1, -1, -\frac{1}{2}\right\rangle\right| = \sqrt{1^2 + (-1)^2 + \left(-\frac{1}{2}\right)^2} = \frac{3}{2}$

   $\theta = \arccos\left(\frac{\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}}{|\tilde{\mathbf{u}}||\tilde{\mathbf{v}}|}\right) = \arccos\left(\frac{\left(\frac{1}{2}\right)}{\left(\sqrt{29}\right)\left(\frac{3}{2}\right)}\right) = \arccos\left(\frac{1}{\left(\sqrt{29}\right)(3)}\right)$

   $$\boxed{\theta = \arccos\left(\frac{1}{\left(\sqrt{29}\right)(3)}\right)}$$