MTH 263 Practice Test #1

SPRING 1999

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1. Find the area of the region bounded by the graph $r = 2a\cos(\theta)$.

   Observe: This is a circle of radius a, for

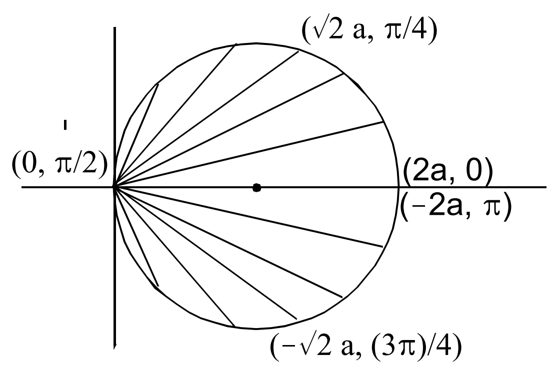
   $r = 2a\cos(\theta) \Rightarrow r = 2a\left(\frac{x}{r}\right) \Rightarrow r^2 = 2ax \Rightarrow x^2 + y^2 = 2ax \Rightarrow x^2 - 2ax + y^2 = 0$

   $\Rightarrow x^2 - 2ax + \left(\frac{-2a}{2}\right)^2 + y^2 = \left(\frac{-2a}{2}\right)^2 \Rightarrow (x-a)^2 + y^2 = a^2$

   This is a circle of radius a centered at $(a, 0)$.

   To get a better idea of how the graph is laid out, we'll plot a few values of r vs θ.

θ	r
0	$2a$
$\frac{\pi}{4}$	$\sqrt{2}a$
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	$-\sqrt{2}a$
π	$-2a$



$A = \frac{1}{2}\int_{\theta_a}^{\theta_b} [f(\theta)]^2 d\theta = \frac{1}{2}\int_0 \pi [2a\cos(\theta)]^2 d\theta = \frac{1}{2}\int_0 \pi 4a^2\cos^2(\theta)\, d\theta = 2a^2\int_0 \pi \cos^2(\theta)\, d\theta =$
$2a^2\int_0 \pi \frac{1+\cos(2\theta)}{2} d\theta = a^2\int_0 \pi (1+\cos(2\theta))\, d\theta =$
$a^2\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0 \pi = \pi a^2.$

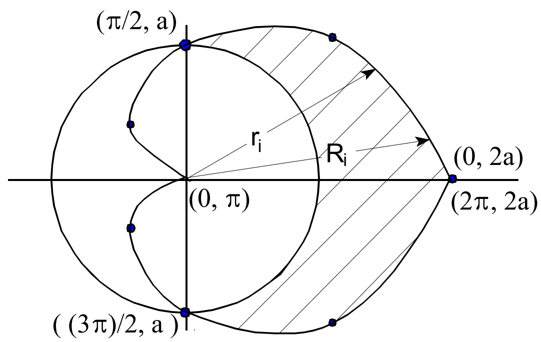
Note that this is what we should have come up with, considering that the region was a circle of radius a.

More importantly, note that we didn't just blindly integrate from 0 to 2π. We had to consider the "endpoints", or rays of the **angle** that we partitioned into sub-angles. Here, we partitioned the angle with initial and terminal rays 0 and π into sub-angles. Hence, the limits of the integral are 0 and π.

2. Find the area of the region that is inside the "cardioid" $r = a(1+\cos(\theta))$, but outside the circle $r = a$.

   We create a table of values r vs θ.

θ	r
0	$2a$
$\frac{\pi}{4}$	$a+\frac{\sqrt{2}a}{2}$
$\frac{\pi}{2}$	a
$\frac{3\pi}{4}$	$a-\frac{\sqrt{2}a}{2}$
π	0
$\frac{5\pi}{4}$	$a-\frac{\sqrt{2}a}{2}$
$\frac{3\pi}{2}$	a
$\frac{7\pi}{4}$	$a+\frac{\sqrt{2}a}{2}$
2π	$2a$



To do this problem, observe that:
Area i^{th} "subregion" = Area i^{th} large sector - Area i^{th} small sector

Area of i^{th} large sector $= \frac{\triangle\theta}{2\pi}\pi R_i^2 = \frac{1}{2}R_i^2 \triangle\theta$

Area of i^{th} small sector $= \frac{\triangle\theta}{2\pi}\pi r_i^2 = \frac{1}{2}r_i^2 \triangle\theta$

Area of i^{th} subregion $= \frac{1}{2}R_i^2 \triangle\theta - \frac{1}{2}r_i^2 \triangle\theta = \frac{1}{2}\left(R_i^2 - r_i^2\right)\triangle\theta$

Area total $= \sum_{i=1}^{n}\frac{1}{2}\left(R_i^2 - r_i^2\right)\triangle\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(R^2-r^2\right)d\theta + \frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi}\left(R^2-r^2\right)d\theta =$

$\frac{1}{2}\int_0^{\frac{\pi}{2}}\left([a(1+\cos(\theta))]^2 - a^2\right)d\theta + \frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi}\left([a(1+\cos(\theta))]^2 - a^2\right)d\theta =$

$\frac{1}{2}a^2\int_0^{\frac{\pi}{2}}\left[2\cos(\theta)+\cos^2(\theta)\right]d\theta + \frac{1}{2}a^2\int_{\frac{3\pi}{2}}^{2\pi}\left[2\cos(\theta)+\cos^2(\theta)\right]d\theta =$

$\frac{1}{2}a^2\int_0^{\frac{\pi}{2}}\left[2\cos(\theta)+\frac{1+\cos(2\theta)}{2}\right]d\theta + \frac{1}{2}a^2\int_{\frac{3\pi}{2}}^{2\pi}\left[2\cos(\theta)+\frac{1+\cos(2\theta)}{2}\right]d\theta =$

$\frac{1}{2}a^2\left[2\sin(\theta)+\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right]_0^{\frac{\pi}{2}} + \frac{1}{2}a^2\left[2\sin(\theta)+\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right]_{\frac{3\pi}{2}}^{2\pi} =$

$\frac{1}{2}a^2\left[\left(2+\frac{\pi}{4}\right)-(0)\right] + \frac{1}{2}a^2\left[\pi - \left(-2+\frac{3\pi}{4}\right)\right] = 2a^2 + \frac{\pi a^2}{4}$

3. Find the area bounded by the graph of $r^2 = 4\sin(2\theta)$.

Observe: As θ increases from 0 to π, the entire graph is generated, as any two values of θ that differ by a whole number multiple of π yield the same value of r^2.

e.g. Suppose $\theta_b = \theta_a + \pi$.

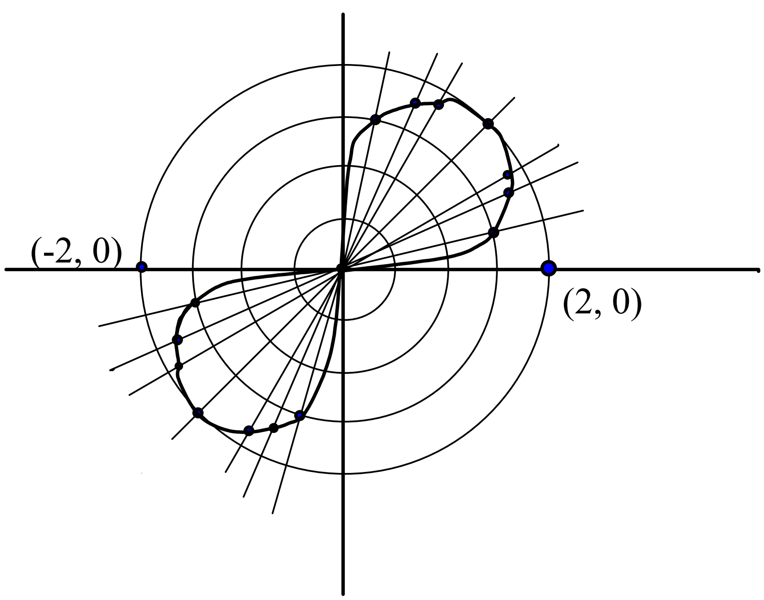
Then $r_b^2 = 4\sin(2\theta_b) = 4\sin(2(\theta_a + \pi)) = 4\sin(2\theta_a + 2\pi) = 4\sin(2\theta_a) = r_a^2$.

So θ between 0 and π generates the entire graph.

Note Also: For $\frac{\pi}{2} < \theta < \pi$ we have $r^2 < 0$, which is impossible for real values of r. Therefore, θ between $\frac{\pi}{2}$ and π generates nothing - the entire graph is generated by θ between 0 and $\frac{\pi}{2}$.

We plot some values of r vs θ.

θ	r^2	r
0	0	0
$\frac{\pi}{12}$	2	$\pm\sqrt{2}$
$\frac{\pi}{8}$	$2\sqrt{2}$	$\pm\sqrt[4]{8}$
$\frac{\pi}{6}$	$2\sqrt{3}$	$\pm\sqrt[4]{12}$
$\frac{\pi}{4}$	4	± 2
$\frac{\pi}{3}$	$2\sqrt{3}$	$\pm\sqrt[4]{12}$
$\frac{3\pi}{8}$	$2\sqrt{2}$	$\pm\sqrt[4]{8}$
$\frac{5\pi}{12}$	2	$\pm\sqrt{2}$
$\frac{\pi}{2}$	0	0



Area of one petal $= \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 4\sin(2\theta)\,d\theta = 2\int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta = -\cos(2\theta)|_0^{\frac{\pi}{2}} = -\cos(\pi) - (-\cos(0)) = 2$

Total area is the area of two petals which is 4.

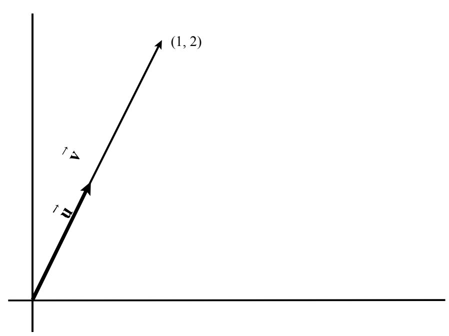
4. Compute the norm of $\vec{\mathbf{v}} = \langle 1, 3, 5 \rangle$
   $||\vec{\mathbf{v}}|| = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}$
   Alternately, $||\vec{\mathbf{v}}|| = \sqrt{\vec{\mathbf{v}} \circ \vec{\mathbf{v}}} = \sqrt{1 \cdot 1 + 3 \cdot 3 + 5 \cdot 5} = \sqrt{35}$.

5. Find a unit vector in $\Re^3$ having the same direction as $\vec{\mathbf{v}} = \langle 1, 3, 5 \rangle$.

   From our previous definition, this would be
   $\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{v}}||} = \frac{\langle 1,3,5 \rangle}{\sqrt{35}} = \left\langle \frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}} \right\rangle$.

6. Find a unit vector in $\Re^2$ having the same direction as $\vec{\mathbf{v}} = \langle 1, 2 \rangle$.

   $\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{v}}||} = \frac{\langle 1,2 \rangle}{\sqrt{1^2+2^2}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$. The geometric relationship between the two is seen in the picture below:



7. Given vectors $\vec{\mathbf{u}} = \langle 1, 2 \rangle$ and $\vec{\mathbf{v}} = \langle 4, 3 \rangle$, compute the angle θ between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

   $\cos(\theta) = \frac{\vec{\mathbf{u}} \circ \vec{\mathbf{v}}}{||\vec{\mathbf{u}}||||\vec{\mathbf{v}}||} = \frac{4+6}{\sqrt{5}\sqrt{25}} = \frac{10}{5\sqrt{5}} = \frac{2}{\sqrt{5}} \Longrightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$.

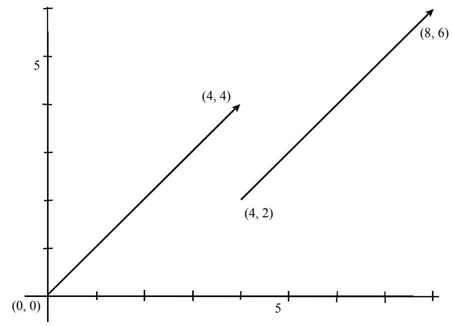
8. Given vectors $\vec{\mathbf{u}} = \langle 1, 2 \rangle$ and $\vec{\mathbf{v}} = \langle 4, 3 \rangle$, compute the projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$, $\mathbf{proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}$, and the projection of $\vec{\mathbf{u}}$ orthogonal to $\vec{\mathbf{v}}$, $\mathbf{orth}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}$.

   $\mathbf{proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \frac{\vec{\mathbf{u}} \circ \vec{\mathbf{v}}}{\vec{\mathbf{v}} \circ \vec{\mathbf{v}}}\vec{\mathbf{v}} = \frac{4+6}{25}\langle 4, 3 \rangle = \left\langle \frac{8}{5}, \frac{6}{5} \right\rangle$

   $\mathbf{orth}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \vec{\mathbf{u}} - \mathbf{proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \langle 1, 2 \rangle - \left\langle \frac{8}{5}, \frac{6}{5} \right\rangle = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$

9. Show that the vector $\vec{\mathbf{v}}$ with initial point $(4, 2)$ and terminal point $(8, 6)$ is parallel to the vector $\vec{\mathbf{u}}$ whose initial point is $(0, 0)$ and whose terminal point $(4, 4)$.

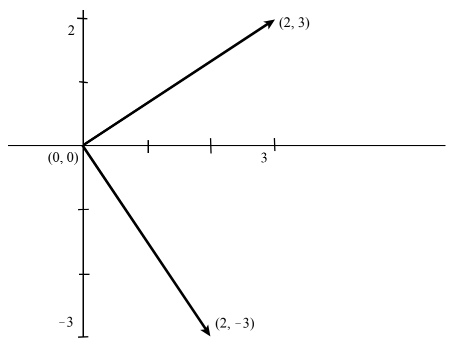
   The vectors are pictured below:



The "component form" of the vector **v** can be computed by subtracting the coordinates of the initial points from the coordinates of the terminal points. Hence, $\vec{\mathbf{v}} = \langle 8-4, 6-2\rangle = \langle 4,4\rangle$. If we compute the component form of $\vec{\mathbf{u}}$ we find that $\vec{\mathbf{u}} = \langle 4-0, 4-0\rangle = \langle 4,4\rangle = \vec{\mathbf{v}}$.

10. Show that the vectors $\vec{\mathbf{u}} = \langle 3,2\rangle$ and $\vec{\mathbf{v}} = \langle 2,-3\rangle$ are perpendicular.

    The vectors are shown below:



$\vec{\mathbf{u}} \circ \vec{\mathbf{v}} = \langle 3,2\rangle \circ \langle 2,-3\rangle = 6-6 = 0$. Since $\vec{\mathbf{u}} \circ \vec{\mathbf{v}} = 0$, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are perpendicular.

11. Let $\vec{\mathbf{v}} = 3\vec{i} - 6\vec{j} + 4\vec{k}$ and $\vec{\mathbf{u}} = \vec{i} + 4\vec{j} + 4\vec{k}$.

    (a) Compute $||\vec{\mathbf{v}}||$

    $||\vec{\mathbf{v}}|| = ||\langle 3,-6,4\rangle|| = \sqrt{3^2 + (-6)^2 + 4^2} = \sqrt{61}$

    (b) Compute $||3\vec{\mathbf{v}} - \vec{\mathbf{u}}||$

    $||3\vec{\mathbf{v}} - \vec{\mathbf{u}}|| = ||3\langle 3,-6,4\rangle - \langle 1,4,4\rangle|| = ||\langle 9,-18,12\rangle - \langle 1,4,4\rangle|| =$
    $||\langle 9-1,-18-4,12-4\rangle|| = ||\langle 8,-22,8\rangle|| = \sqrt{8^2 + (-22)^2 + 8^2} = \sqrt{612} =$
    $\sqrt{(36)(17)} = 6\sqrt{17}$

12. Find the angle between the line containing the points $(1,2,3)$ and $(2,4,6)$, and the line containing the points $(1,2,3)$ and $(3,6,9)$.

    The line containing the points $(1,2,3)$ and $(2,4,6)$ has the same direction as the vector with initial point $(1,2,3)$ and terminal point $(2,4,6)$. In component form, this is the vector $\langle 2-1, 4-2, 6-3 \rangle = \langle 1,2,3 \rangle$.

    Similarly, the line containing the points $(1,2,3)$ and $(3,6,9)$ has the same direction as the vector with initial point $(1,2,3)$ and terminal point $(3,6,9)$. In component form, this vector is $\langle 2,4,6 \rangle$.

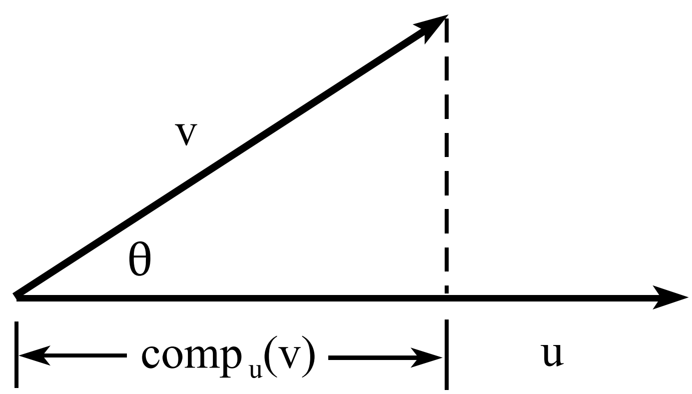
    The angle between the two vectors can be found from the equation

$$\begin{aligned} \vec{\mathbf{u}} \circ \vec{\mathbf{v}} &= ||\vec{\mathbf{u}}||\,||\vec{\mathbf{v}}|| \cos(\theta) \\ &\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{\mathbf{u}} \circ \vec{\mathbf{v}}}{||\vec{\mathbf{u}}||\,||\vec{\mathbf{v}}||}\right) \end{aligned}$$

    Thus we have: $\theta = \cos^{-1}\left(\frac{\langle 1,2,3\rangle \circ \langle 2,4,6\rangle}{||\langle 1,2,3\rangle||||\langle 2,4,6\rangle||}\right) = \cos^{-1}\left(\frac{2+8+18}{\sqrt{1^2+2^2+3^2}\sqrt{2^2+4^2+6^2}}\right) = \cos^{-1}\left(\frac{28}{\sqrt{14}\sqrt{56}}\right) = \cos^{-1}\left(\frac{28}{\sqrt{142}\sqrt{14}}\right) = \cos^{-1}(1) = 0.$

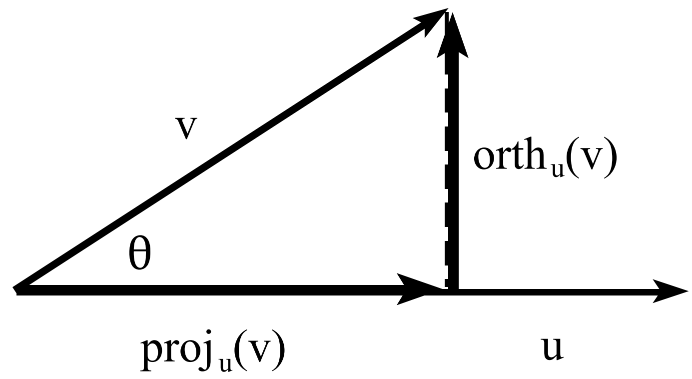
13. Given that $\vec{\mathbf{u}} = \langle 3,-1,2 \rangle$ and $\vec{\mathbf{v}} = \langle 4,1,-5 \rangle$, compute:

    (a) $comp_u(\vec{\mathbf{v}})$



    Observe: $comp_u(\vec{\mathbf{v}}) = ||\vec{\mathbf{v}}|| \cos(\theta)$.
    Also $\vec{\mathbf{u}} \circ \vec{\mathbf{v}} = ||\vec{\mathbf{u}}||\,||\vec{\mathbf{v}}|| \cos(\theta) \Rightarrow \cos(\theta) = \frac{\vec{\mathbf{u}} \circ \vec{\mathbf{v}}}{||\vec{\mathbf{u}}||||\vec{\mathbf{v}}||}$
    $\Rightarrow comp_u(\vec{\mathbf{v}}) = ||\vec{\mathbf{v}}|| \cos(\theta) = ||\vec{\mathbf{v}}|| \frac{\vec{\mathbf{u}} \circ \vec{\mathbf{v}}}{||\vec{\mathbf{u}}||||\vec{\mathbf{v}}||} = \frac{\vec{\mathbf{u}} \circ \vec{\mathbf{v}}}{||\vec{\mathbf{u}}||} = \frac{\langle 3,-1,2\rangle \circ \langle 4,1,-5\rangle}{||\langle 3,-1,2\rangle||} = \frac{12-1-10}{\sqrt{3^2+(-1)^2+2^2}} = \frac{1}{\sqrt{14}}$

    (b) $proj_u(\vec{\mathbf{v}})$
    As for $proj_u(\vec{\mathbf{v}})$, note that $proj_u(\vec{\mathbf{v}})$ is the vector which has the magnitude of $comp_u(\vec{\mathbf{v}})$ and the direction of $\vec{\mathbf{u}}$. (See below.)

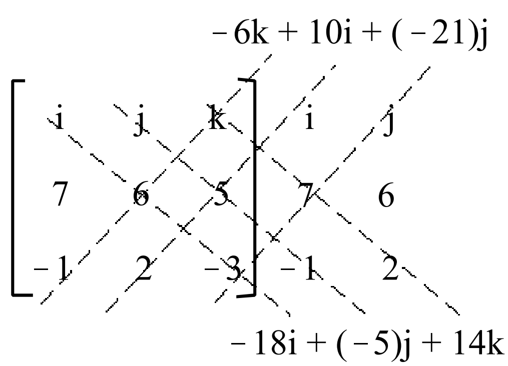


Therefore, to get $proj_u(\vec{\mathbf{v}})$, we'll take its magnitude, $comp_u(\vec{\mathbf{v}})$, and multiply it by the unit vector having the direction of $\vec{\mathbf{u}}$, to give $proj_u(\vec{\mathbf{v}})$ its direction.
$proj_u(\vec{\mathbf{v}}) = comp_u(\vec{\mathbf{v}})\frac{\vec{\mathbf{u}}}{||\vec{\mathbf{u}}||} = \frac{1}{\sqrt{14}}\frac{\langle 3,-1,2\rangle}{\sqrt{3^2+(-1)^2+2^2}} = \frac{1}{\sqrt{14}}\frac{\langle 3,-1,2\rangle}{\sqrt{14}} = \left\langle \frac{3}{14}, -\frac{1}{14}, \frac{1}{7}\right\rangle$.

(c) and $orth_u(\vec{\mathbf{v}})$.
From the picture above, note that $\vec{\mathbf{v}} = proj_u(\vec{\mathbf{v}}) + orth_u(\vec{\mathbf{v}})$
$\Rightarrow orth_u(\vec{\mathbf{v}}) = \vec{\mathbf{v}} - proj_u(\vec{\mathbf{v}}) = \langle 4,1,-5\rangle - \left\langle \frac{3}{14}, -\frac{1}{14}, \frac{1}{7}\right\rangle = \left\langle \frac{53}{14}, \frac{15}{14}, -\frac{36}{7}\right\rangle$

14. $\vec{\mathbf{a}} = 7\vec{i} + 6\vec{j} + 5\vec{k}$ and $\vec{\mathbf{b}} = -\vec{i} + 2\vec{j} - 3\vec{k}$. Compute $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$.

First, we set up the matrix displayed below, and compute the products going along the diagonals.



$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \left(-18\vec{i} - 5\vec{j} + 14\vec{k}\right) - \left(-6\vec{k} + 10\vec{i} - 21\vec{j}\right) = -28\vec{i} + 16\vec{j} + 20\vec{k}$

15. Find the volume of the parallelopiped with vertices $(1,-3,2)$, $(8,3,7)$, $(0,-1,-1)$, and $(4,2,1)$. (Hint: You may be able to use the results of problem #1.)

The vector with initial point $(1,-3,2)$ and terminal point $(8,3,7)$ is equivalent to $\langle 7,6,5\rangle$.

The vector with initial point $(1,-3,2)$ and terminal point $(0,-1,-1)$ is equivalent to $\langle -1,2,-3\rangle$.

The vector with initial point $(1,-3,2)$ and terminal point $(4,2,1)$ is equivalent to $\langle 3,5,-1\rangle$.

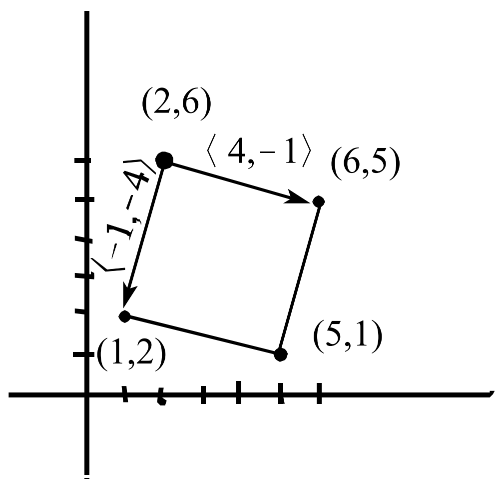
These vectors determine the sides of a parallelopiped.

The volume of the parallelopiped is the triple product:

$$\left\| \langle 3,5,-1\rangle \circ \underbrace{(\langle 7,6,5\rangle \times \langle -1,2,-3\rangle)}_{=\langle -28,16,20\rangle \text{ from prev problem}} \right\|$$

$= ||-84+80-20|| = ||-24|| = 24$

16. Compute the volume of the parallelogram with vertices $(1,2)$, $(2,6)$, and $(6,5)$.



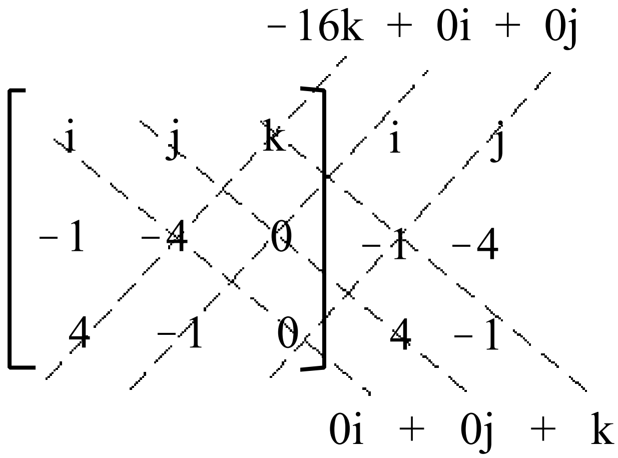
The picture above depicts the situation.

The vector with initial point $(2,6)$ and terminal point $(1,2)$ is equivalent to $\langle -1,-4\rangle$.

The vector with initial point $(2,6)$ and terminal point $(6,5)$ is equivalent to $\langle 4,-1\rangle$.

The area of the parallelogram is $||\langle -1,-4\rangle \times \langle 4,-1\rangle||$.

To compute this, we set up the following matrix:



$= \vec{k} - \left(-16\vec{k}\right) = 17\vec{k}$

The area of the parallelogram is given by: $||\langle -1,-4\rangle \times \langle 4,-1\rangle|| = \left|\left|17\vec{k}\right|\right| = 17$