Multiple Integrals - Part #1 - Solutions

SPRING 2018

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1. Compute: $\int_1^2 \int_{-1}^1 \int_0^2 xyz\, dxdydz$

   The "inner integral" is with respect to x. The "outer integral is with respect to z.

   Thus, we integrate:

   with respect to x first

   with respect to y second

   with respect to z last

   $$\int_1^2 \int_{-1}^1 \int_0^2 xy^2z\, dxdydz = \int_1^2 \int_{-1}^1 \left[\tfrac{1}{2}x^2y^2z\right]_{x=0}^{x=2} dydz = \int_1^2 \int_{-1}^1 \left[\left(\tfrac{1}{2}(2)^2 y^2z\right) - \left(\tfrac{1}{2}(0)^2 y^2z\right)\right] dydz$$

   $$= \int_1^2 \int_{-1}^1 2y^2z\, dydz = \int_1^2 \left[2\left(\tfrac{1}{3}y^3\right)z\right]_{y=-1}^{y=1} dz = \int_1^2 \left[2\left(\tfrac{1}{3}(1)^3\right)z - 2\left(\tfrac{1}{3}(-1)^3\right)z\right] dz$$

   $$= \int_1^2 \tfrac{4}{3}z\, dz = \left[\tfrac{4}{3}\left(\tfrac{1}{2}z^2\right)\right]_{z=1}^{z=2} = \tfrac{4}{3}\left(\tfrac{1}{2}(2)^2\right) - \tfrac{4}{3}\left(\tfrac{1}{2}(1)^2\right) = \tfrac{8}{3} - \tfrac{2}{3} = 2$$

   $$\boxed{\int_1^2 \int_{-1}^1 \int_0^2 xy^2z\, dxdydz = 2}$$

2. Compute: Compute: $\int_0^9 \int_0^{\frac{y}{3}} \int_0^{\sqrt{y^2-9x^2}} z\, dzdxdy$

   The "inner integral" is with respect to z. The "outer integral is with respect to y.

   Thus, we integrate:

   with respect to z first

   with respect to x second

   with respect to y last

   $$\int_0^9 \int_0^{\frac{y}{3}} \int_0^{\sqrt{y^2-9x^2}} z\, dzdxdy = \int_0^9 \int_0^{\frac{y}{3}} \left[\tfrac{1}{2}z^2\right]_0^{\sqrt{y^2-9x^2}} dxdy = \int_0^9 \int_0^{\frac{y}{3}} \left[\left(\tfrac{1}{2}\left(\sqrt{y^2-9x^2}\right)^2\right) - \left(\tfrac{1}{2}(0)^2\right)\right] dxdy$$

   $$= \int_0^9 \int_0^{\frac{y}{3}} \tfrac{1}{2}\left(y^2 - 9x^2\right) dxdy = \tfrac{1}{2}\int_0^9 \left[\left(y^2x - 3x^3\right)\right]_0^{\frac{y}{3}} dy$$

   $$= \tfrac{1}{2}\int_0^9 \left[\left(y^2\left(\tfrac{y}{3}\right) - 3\left(\tfrac{y}{3}\right)^3\right) - \left(y^2(0) - 3(0)^3\right)\right] dy = \tfrac{1}{2}\int_0^9 \left(\tfrac{1}{3}y^3 - \tfrac{1}{9}y^3\right) dy$$

   $$= \tfrac{1}{2}\int_0^9 \tfrac{2}{9}y^3\, dy = \tfrac{1}{9}\int_0^9 y^3\, dy = \tfrac{1}{9}\left[\tfrac{1}{4}y^4\right]_0^9 = \tfrac{1}{9}\left[\tfrac{1}{4}(9)^4 - \tfrac{1}{4}(0)^4\right]_0^9 = \tfrac{1}{4}(9)^3 = \tfrac{729}{4}$$

   $$\boxed{\int_0^9 \int_0^{\frac{y}{3}} \int_0^{\sqrt{y^2-9x^2}} z\, dzdxdy = \tfrac{729}{4}}$$

3. Reverse the order of integration:

$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} f(x,y)\,dydx$

What we have here, is the integral:

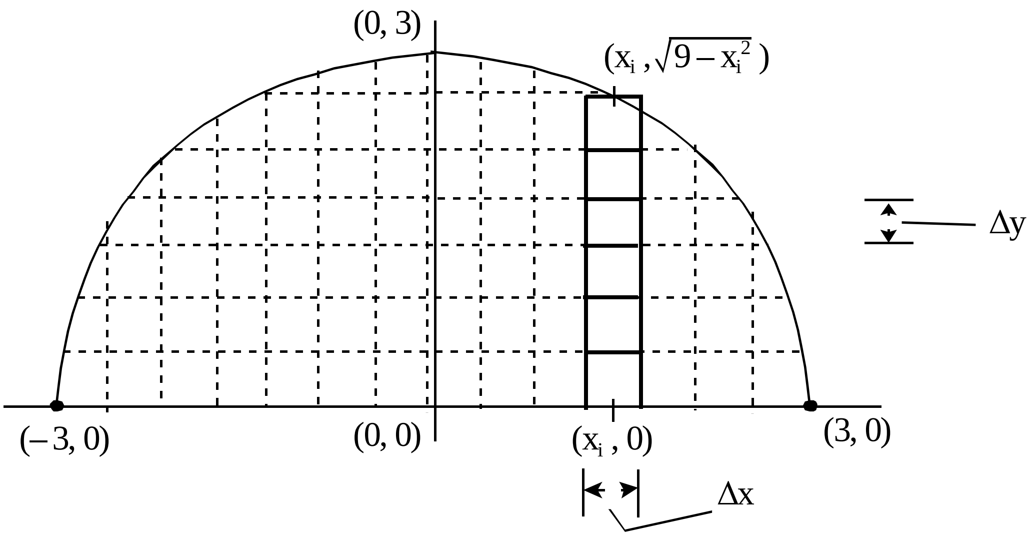
$\int_{x=-3}^{x=3}\int_{y=0}^{y=\sqrt{9-x^2}} f(x,y)\,dydx$

The first thing that we do is sketch the area over which $f(x,y)$ is integrated.

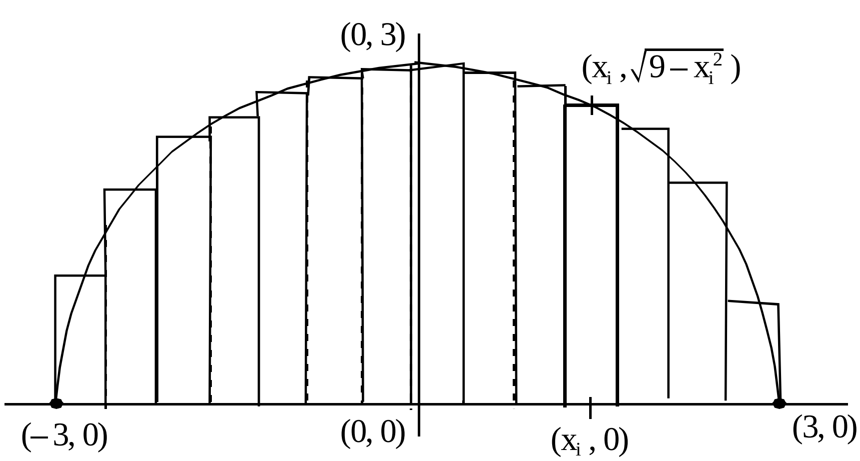
Over each rectangle of area $\Delta x\Delta y$, we compute the value of $f(x,y)$, to get $f(x_i,y_i)\,\Delta x\Delta y$. This is the **volume** of a column of height $f(x_i,y_i)$ and cross-sectional area $\Delta x\Delta y$.

The "inner integral" $\int_{y=0}^{y=\sqrt{9-x^2}} f(x,y)\,dy$ tells us that over the row in which $x = x_i$, we add up the volumes of the columns $f(x_i,y_j)\,\Delta x\Delta y$, from $y = 0$ to $y = \sqrt{9-x^2}$. (Note that the graph of $y = \sqrt{9-x^2}$ is the semi-circle of radius 3, lying above the x-axis.)

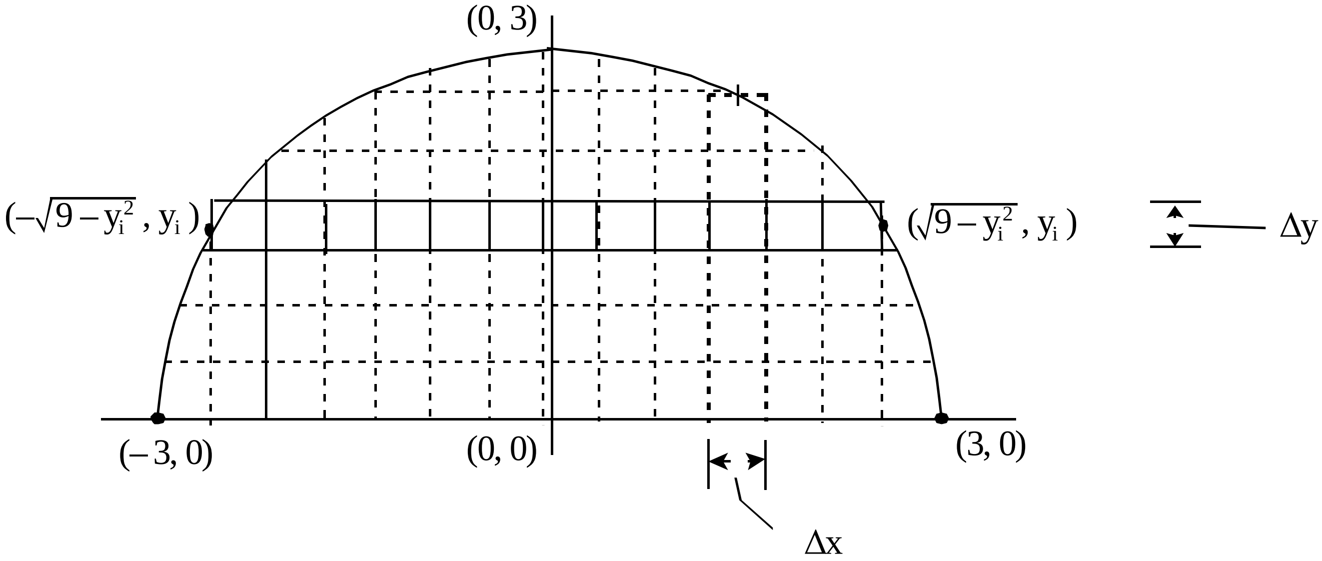
This forms the "i$^{\text{th}}$ slice."



The "outer integral," $\int_{x=-3}^{x=3} dx$, indicates that we add up the volumes of the slices from $x = -3$ to $x = 3$ (See Below)

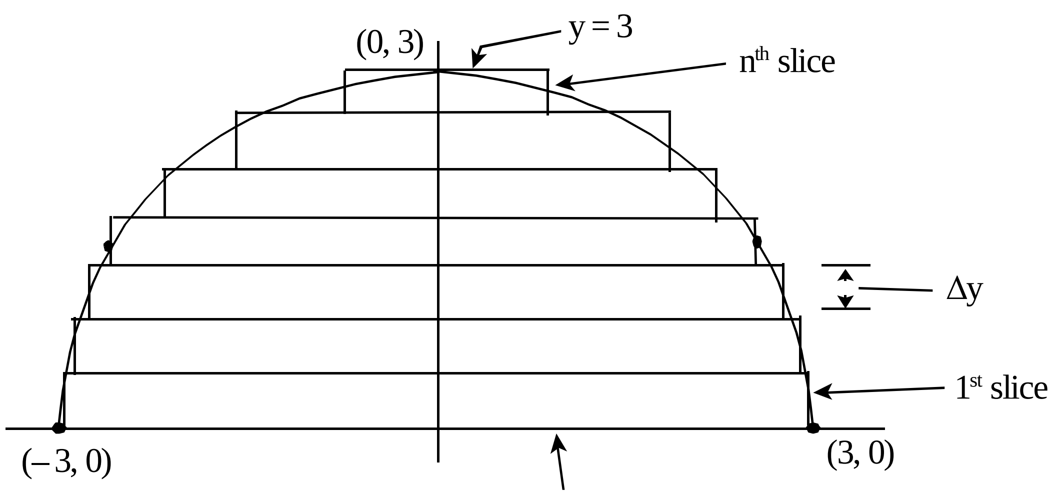


To **reverse the order of integration,** we hold y_j constant, and add up the volumes of the columns in the j^{th} row, from $x = -\sqrt{9 - y_j}$ to $x = \sqrt{9 - y_j}$, forming the "j^{th} slice." (See the picture below.)



This yields the "inner integral" $\int_{x=-\sqrt{9-y_j^2}}^{x=\sqrt{9-y_j^2}} f(x, y_j)\,dx$

To complete the process, we add up the volumes of the slices from $y = 0$ to $y = 3$, yielding the integral $\int_{y=0}^{y=3} \int_{x=-\sqrt{9-y^2}}^{x=\sqrt{9-y^2}} f(x, y)\,dxdy$. (See Below)



The "reversed integral" that we seek is $\int_{y=0}^{y=3} \int_{x=-\sqrt{9-y^2}}^{x=\sqrt{9-y^2}} f(x, y)\,dxdy$