MTH 2227 Test #2

Spring 2018

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Show CLEARLY how you arrive at your answers.

1. A parallelepiped has vertices $(2,1,4)$, $(3,2,5)$, $(2,0,3)$, and $(-1,2,2)$, where the latter vertices given are adjacent to vertex $(2,1,4)$. Compute the volume of the parallelepiped.

   We'll call the vector, with endpoints $(2,1,4)$ and $(3,2,5)$, $\vec{u} = \langle 3-2, 2-1, 5-4 \rangle = \langle 1,1,1 \rangle$

   We'll call the vector, with endpoints $(2,1,4)$ and $(2,0,3)$, $\vec{v} = \langle 2-2, 0-1, 3-4 \rangle = \langle 0,-1,-1 \rangle$

   We'll call the vector, with endpoints $(2,1,4)$ and $(-1,2,2)$, $\vec{w} = \langle (-1)-2, 2-1, 2-4 \rangle = \langle -3,1,-2 \rangle$

   $\vec{u} = \langle 1,1,1 \rangle$

   $\vec{v} = \langle 0,-1,-1 \rangle$

   $\vec{w} = \langle -3,1,-2 \rangle$

   The volume of the parallelepiped defined by these three vectors, is the absolute value of the triple product: $\vec{u} \circ (\vec{v} \times \vec{w})$

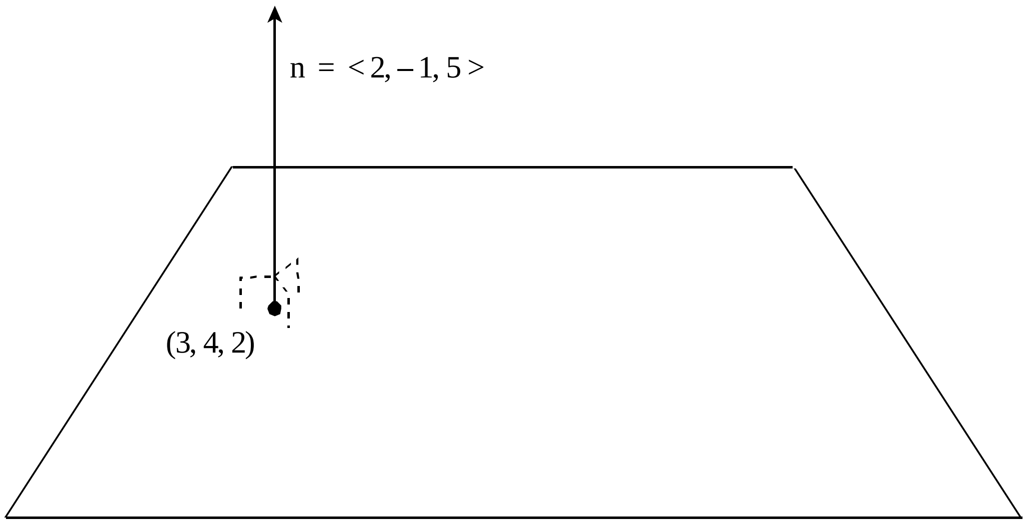
   $$(\vec{v} \times \vec{w}) = \left| \begin{array}{ccc} i & j & k \\ 0 & -1 & -1 \\ -3 & 1 & -2 \end{array} \right| \begin{array}{cc} i & j \\ 0 & -1 \\ -3 & 1 \end{array} = (2i + 3j) - (3k - i) = 3i + 3j - 3k = \langle 3,3,-3 \rangle$$

   $$|\vec{u} \circ (\vec{v} \times \vec{w})| = |\langle 1,1,1 \rangle \circ \langle 3,3,-3 \rangle| = |3+3-3| = 3$$

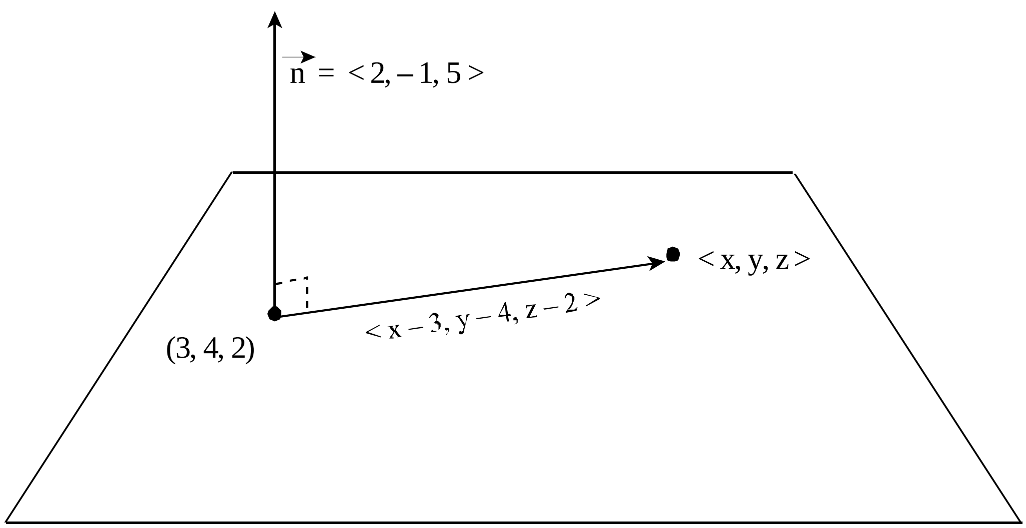
The volume of the parallelepiped is 3

2. Give the parametric equation of the plane that has normal vector $\tilde{\mathbf{n}} = \langle 2, -1, 5 \rangle$ and contains the point $(3, 4, 2)$

   The plane that fits this description is shown below:



If (x, y, z) is any other point in the plane, then $\langle x - 3, y - 4, z - 2 \rangle$ is a vector in the plane that is orthogonal to $\tilde{\mathbf{n}}$. (See below)



Thus, $\tilde{\mathbf{n}} \circ \langle x - 3, y - 4, z - 2 \rangle = 0$

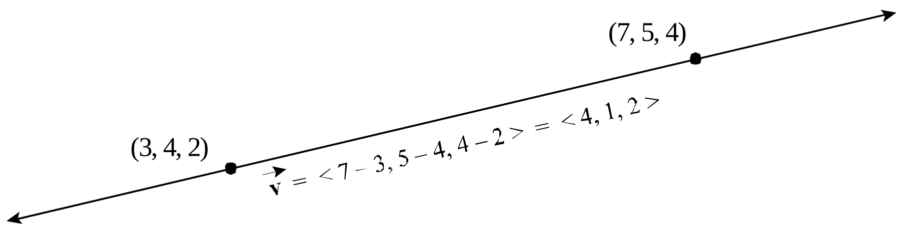
$\Rightarrow \langle 2, -1, 5 \rangle \circ \langle x - 3, y - 4, z - 2 \rangle = 2(x - 3) - 1(y - 4) + 5(z - 2) = 0$

$\Rightarrow 2x - y + 5z = 12$

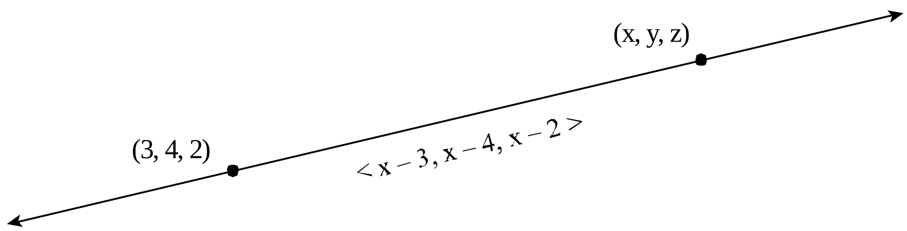
The equation of the plane is $2x - y + 5z = 12$

3. Give the parametric equations of the line that contains the points $(3,4,2)$ and $(7,5,4)$

If $(3,4,2)$ and $(7,5,4)$ are points on a line, then $\tilde{\mathbf{v}} = \langle 7-3, 5-4, 4-2\rangle = \langle 4,1,2\rangle$ is a vector that is parallel to the line. (See below)



If (x,y,z) is any point on the line, then $\langle x-3, y-4, z-2\rangle$ is also a vector that is parallel to the line, and hence, parallel to the vector $\tilde{\mathbf{v}} = \langle 4,1,2\rangle$. (See below)



Thus $\langle x-3, y-4, z-2\rangle$ is parallel to $\tilde{\mathbf{v}} = \langle 4,1,2\rangle$

$\Rightarrow \langle x-3, y-4, z-2\rangle = t\,\langle 4,1,2\rangle = \langle 4t, t, 2t\rangle$

i.e., $\langle x-3, y-4, z-2\rangle = \langle 4t, t, 2t\rangle$

$$\Rightarrow \begin{array}{l} x-3=4t \\ y-4=t \\ z-2=2t \end{array} \quad \Rightarrow \boxed{\begin{array}{l} x=4t+3 \\ y=t+4 \\ z=2t+2 \end{array}}$$

4. Compute: $\int_0^{\frac{\pi}{2}} \int_0^{\frac{y}{2}} \int_0^{\frac{1}{y}} \sin(y)\ dzdxdy$

$$\int_{y=0}^{y=\frac{\pi}{2}} \int_{x=0}^{x=\frac{y}{2}} \int_{z=0}^{z=\frac{1}{y}} \sin(y)\ dzdxdy = \int_{y=0}^{y=\frac{\pi}{2}} \int_{x=0}^{x=\frac{y}{2}} \left[\sin(y)\, z\right]_{z=0}^{z=\frac{1}{y}}\ dxdy$$

$$= \int_{y=0}^{y=\frac{\pi}{2}} \int_{x=0}^{x=\frac{y}{2}} \left[\sin(y)\left(\frac{1}{y}\right) - \sin(y)\,(0)\right]\ dxdy = \int_{y=0}^{y=\frac{\pi}{2}} \int_{x=0}^{x=\frac{y}{2}} \sin(y)\left(\frac{1}{y}\right)\ dxdy$$

$$= \int_{y=0}^{y=\frac{\pi}{2}} \left[\sin(y)\left(\frac{1}{y}\right) x\right]_{x=0}^{x=\frac{y}{2}}\ dy = \int_{y=0}^{y=\frac{\pi}{2}} \left[\sin(y)\left(\frac{1}{y}\right)\left(\frac{y}{2}\right) - \sin(y)\left(\frac{1}{y}\right)(0)\right]\ dy$$

$$= \frac{1}{2}\int_{y=0}^{y=\frac{\pi}{2}} \sin(y)\ dy = \frac{1}{2}\left[-\cos(y)\right]_{y=0}^{y=\frac{\pi}{2}} = \frac{1}{2}\left[-\cos\left(\frac{\pi}{2}\right) - \left(-\cos(0)\right)\right] = \frac{1}{2}\left[0+1\right] = \frac{1}{2}$$

$$\boxed{\int_0^{\frac{\pi}{2}} \int_0^{\frac{y}{2}} \int_0^{\frac{1}{y}} \sin(y)\ dzdxdy = \frac{1}{2}}$$

5. Use a triple integral to compute the volume of the region (in the first octant) bounded by the graphs of $2x + 4y + z = 8$; $z = 0$; $x = 0$; and $y = 0$

We can find the intercepts of the surface $2x + 4y + z = 8$ with the coordinate axes by setting each pair of variables equal to zero.

$\boxed{x\text{-axis } (y = 0,\ z = 0)} \Rightarrow 2x + 4(0) + (0) = 8 \Rightarrow 2x = 8 \Rightarrow x = 4$ (x-intercept)

$\boxed{y\text{-axis } (x = 0,\ z = 0)} \Rightarrow 2(0) + 4y + (0) = 8 \Rightarrow 4y = 8 \Rightarrow y = 2$ (y-intercept)

$\boxed{z\text{-axis } (x = 0,\ y = 0)} \Rightarrow 2(0) + 4(0) + z = 8 \Rightarrow z = 8$ (z-intercept)

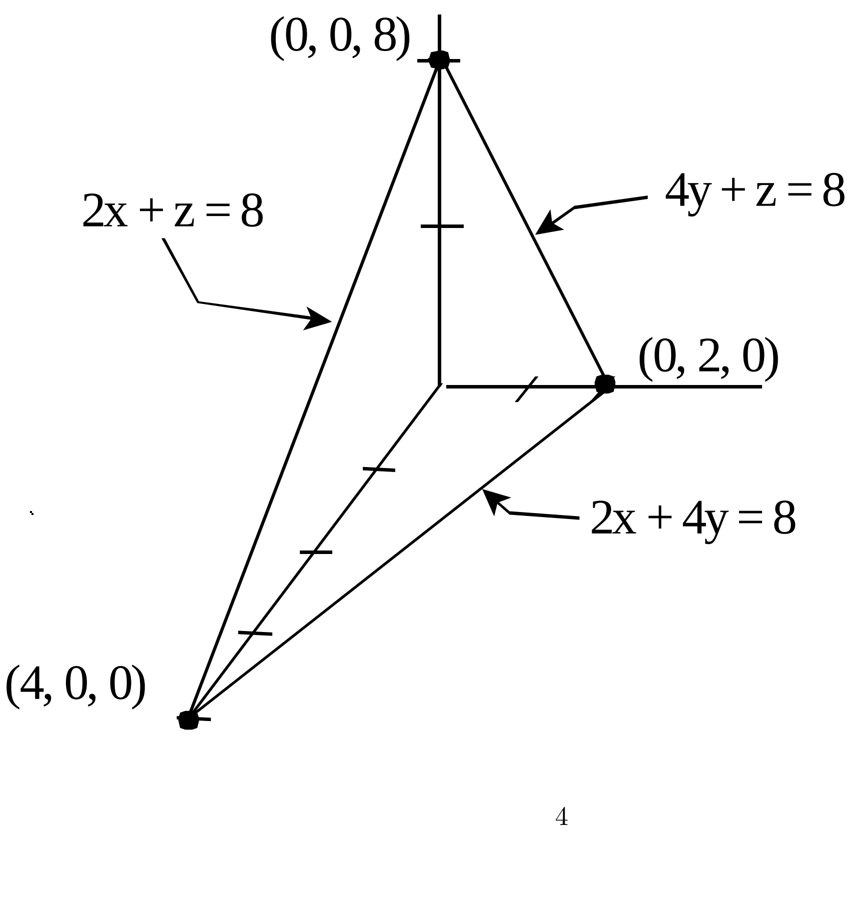
The intersection of the surface $2x + 4y + z = 8$ with the coordinate planes can be found by setting each variable equal to zero.

$\boxed{x\text{-}y \text{ plane } (z = 0)} \Rightarrow 2x + 4y + (0) = 8 \Rightarrow 2x + 4y = 8$

$\boxed{x\text{-}z \text{ plane } (y = 0)} \Rightarrow 2x + 4(0) + z = 8 \Rightarrow 2x + z = 8$

$\boxed{y\text{-}z \text{ plane } (x = 0)} \Rightarrow 2(0) + 4y + z = 8 \Rightarrow 4y + z = 8$

The bounded region is shown in the illustration below.



If we integrate with respect to z first, x second, and y last, then:

The "floor" of the region is the surface $z = 0$ (the x-y plane)

The "ceiling" of the region is the surface $2x + 4y + z = 8$ (i.e., $z = 8 - 2x - 4y$)

The back side is the surface $x = 0$ (the y-z plane)

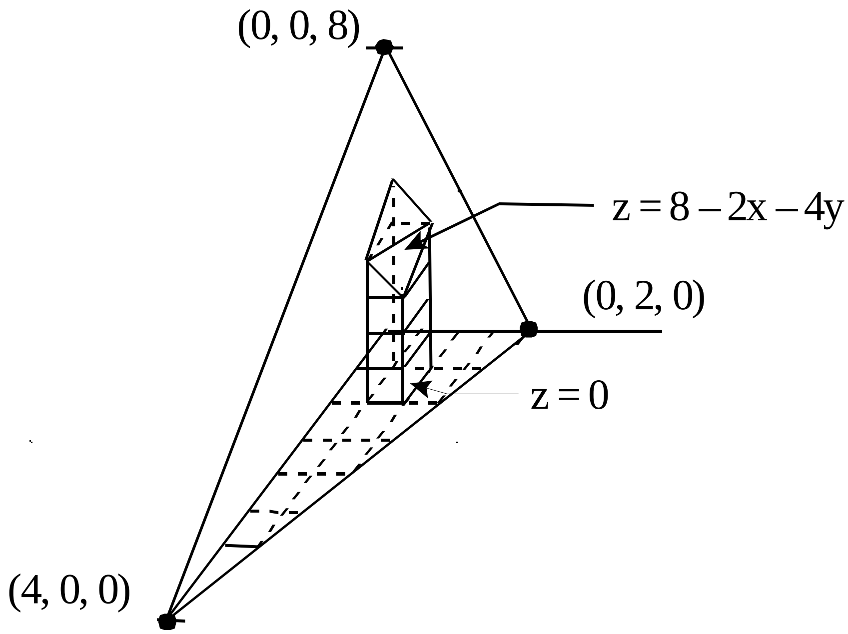
The front side is the surface $2x + 4y + z = 8$ (i.e., $x = -2y - \frac{1}{2}z + 4$)

The left side is the surface $y = 0$ (the x-z plane)

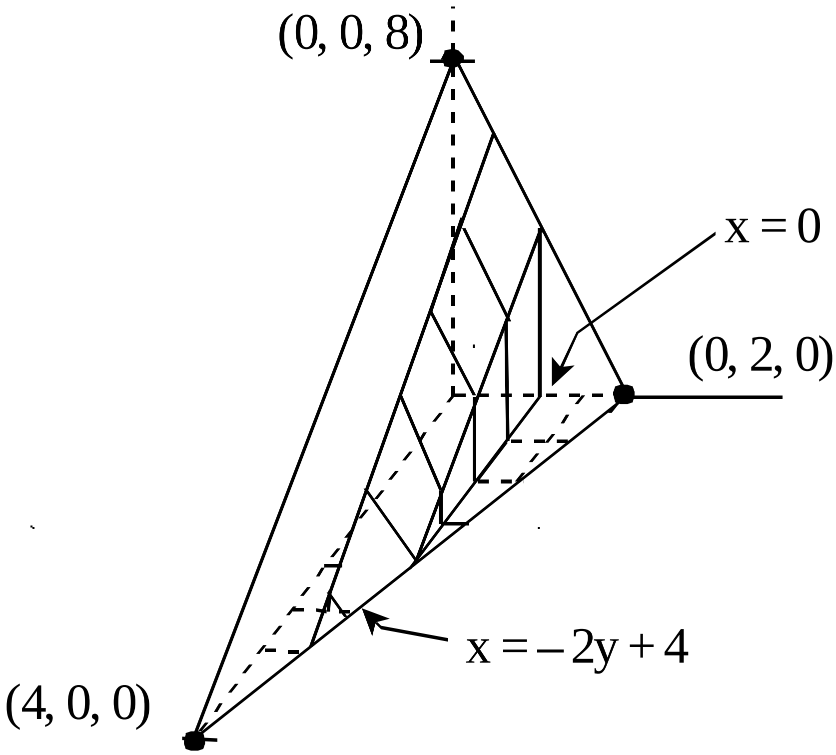
The right side is the surface $2x + 4y + z = 8$ (i.e., $y = -\frac{1}{2}x - \frac{1}{4}z + 2$)

Integrating with respect to z

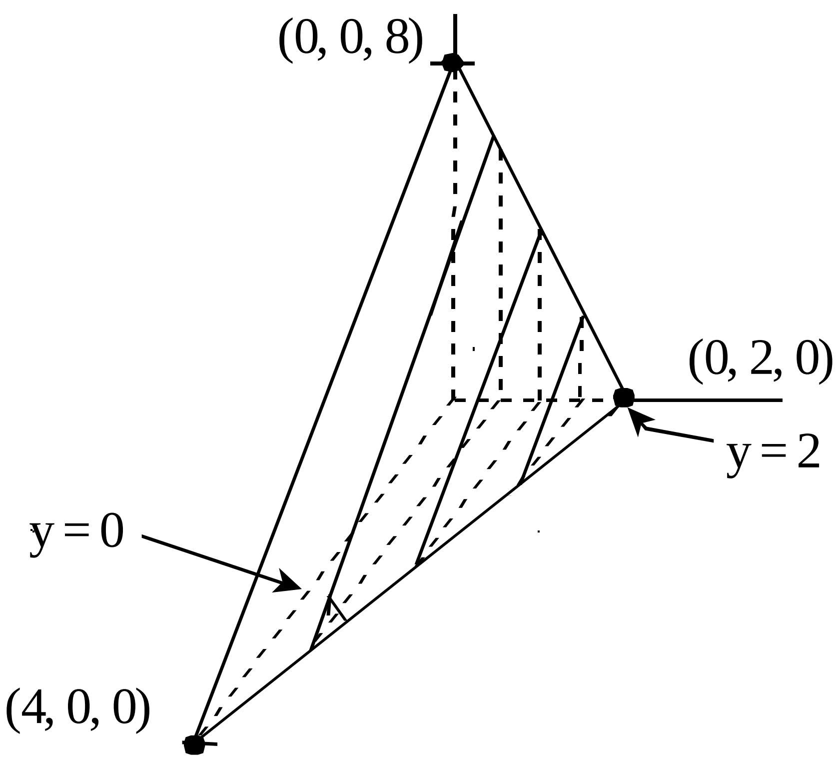
We adding cubes of volume $\Delta z \Delta y \Delta x$ from the floor $z = 0$ to the ceiling $z = 8 - 2x - 4y$ (Shown below).



Next, integrating in the x-direction, we add up the columns of cross-sectional area $\Delta y \Delta x$, from the line $x = 0$ to the line $2x + 4y = 8$ (i.e., $x = -2y + 4$). (Shown below)



Finally, we integrate in the y-direction, adding up the volumes of the slices of thickness Δy from $y = 0$ to $y = 2$ (See below)



This yields the integral:

$\int_{y=0}^{y=2} \int_{x=0}^{x=-2y+4} \int_{z=0}^{z=8-2x-4y} 1 dz dx dy = \int_{y=0}^{y=2} \int_{x=0}^{x=-2y+4} [z]_{z=0}^{z=8-2x-4y} dx dy$

$= \int_{y=0}^{y=2} \int_{x=0}^{x=-2y+4} [(8-2x-4y)-(0)]\, dx dy = \int_{y=0}^{y=2} \int_{x=0}^{x=-2y+4} (8-2x-4y)\, dx dy$

$= \int_{y=0}^{y=2} \left[8x - x^2 - 4yx\right]_{x=0}^{x=-2y+4} dy$

$= \int_{y=0}^{y=2} \left[\left(8(-2y+4) - (-2y+4)^2 - 4y(-2y+4)\right) - \left(8(0) - (0)^2 - 4y(0)\right)\right] dy$

$= \int_{y=0}^{y=2} \left(4y^2 - 16y + 16\right) dy = \left[\frac{4}{3}y^3 - 8y^2 + 16y\right]_{y=0}^{y=2}$

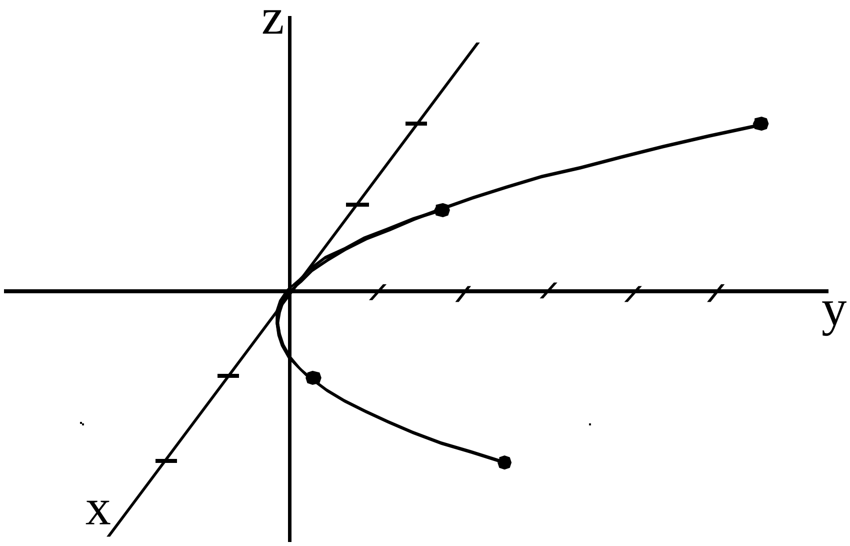
$= \left(\frac{4}{3}(2)^3 - 8(2)^2 + 16(2)\right) - \left(\frac{4}{3}(0)^3 - 8(0)^2 + 16(0)\right) = \frac{32}{3}$

Volume $= \frac{32}{3}$

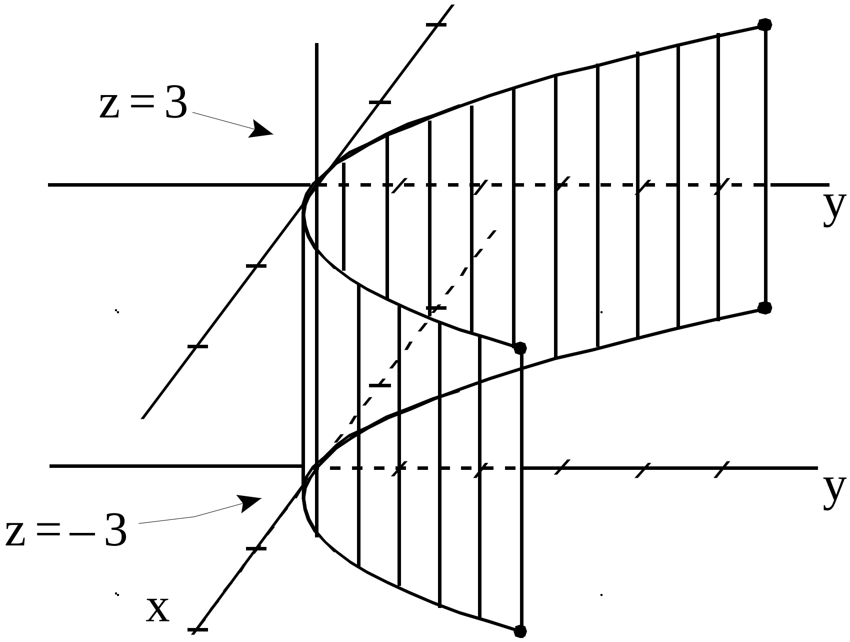
6. Graph the 3-Dimensional surface given by: $y = x^2$

   This is a three dimensional surface lacking the variable z. This means that this is a cylinder. (The cross section of the surface is the same for all values of z.)

   The graph of $y = x^2$ is shown in the plane $z = 0$



   We sketch the same graph for other values of z - let's say $z = \pm 3$, draw the visible lines of the surface, and then draw some lighter lines to give the surface some definition. (Next Page)


z = 3
y
y
z = – 3
x

Extra! (10 points - WOW!) Reverse the order of integration and compute the integral:

$\int_0^1 \int_x^{\sqrt{x}} 2y \, dydx = \frac{1}{6}$

$\int_0^1 \int_{y^2}^{y} 2y \, dxdy = \frac{1}{6}$

Extra! (10 points - WOW!) Reverse the order of integration and compute the integral:

$\int_0^1 \int_x^{\sqrt{x}} 2y \, dydx = \frac{1}{6}$

$\int_0^1 \int_{-x+1}^{1-x^2} 1 \, dydx = \frac{1}{6}$