MTH 263 Practice Test #3 - Solutions

SPRING 1999

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1. Sketch the space curve generated by the vector-valued function $r(t) = 3\cos(t)\vec{\mathbf{i}} + 4\sin(t)\vec{\mathbf{j}} + \frac{1}{2}t\vec{\mathbf{k}}$.

   Examining the x and y components only, we have:

$$x = 3\cos(t); \quad y = 4\sin(t)$$

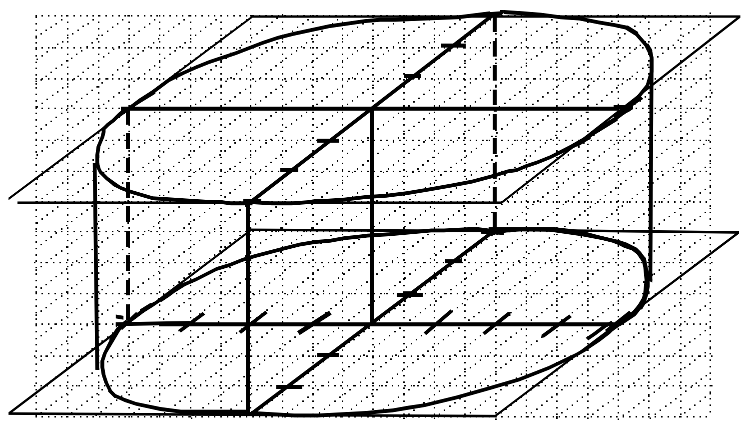
   $\Rightarrow \cos(t) = \frac{x}{3} \quad \sin(t) = \frac{y}{4}$

   $\Rightarrow \cos^2(t) = \frac{x^2}{3^2} \quad \sin^2(t) = \frac{y^2}{4^2}$

   From the identity $\cos^2(t) + \sin^2(t) = 1$, we get

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

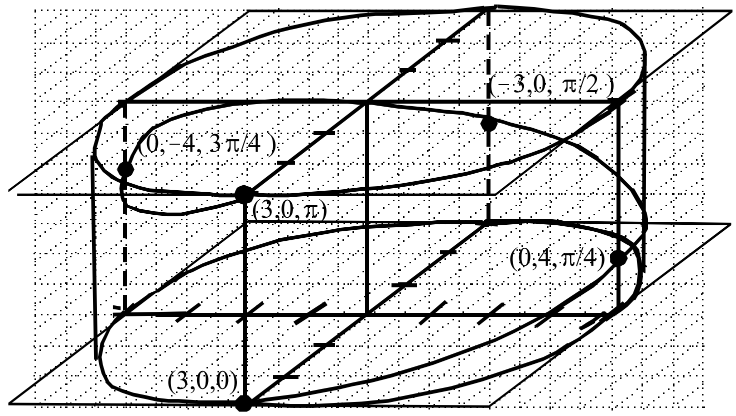
   This is an ellipse, centered at the origin, with "horizontal radius" $a = 3$, and "vertical radius" $b = 4$. In 3-D, this an elliptic cylinder (shown below).



   This surface is not generated by the vector function, $r(t)$, but the space curve generated by $r(t)$ lies on the inside wall of the cylinder. To graph the space curve, we'll plot a few points.

t	$x = 3\cos(t)$	$y = 4\sin(t)$	$z = \frac{1}{2}t$
0	3	0	0
$\frac{\pi}{2}$	0	4	$\frac{\pi}{4}$
π	-3	0	$\frac{\pi}{2}$
$\frac{3\pi}{2}$	0	-4	$\frac{3\pi}{4}$
2π	3	0	π

   The graph of the space curve appears below.



2. Compute the velocity and acceleration vectors for the vector in problem 1, evaluate and graph these vectors at $t = \frac{\pi}{2}$

   $\mathbf{v}(t) = \mathbf{r}'(t) = -3\sin(t)\,\vec{\mathbf{i}} + 4\cos(t)\,\vec{\mathbf{j}} + \frac{1}{2}\vec{\mathbf{k}}$

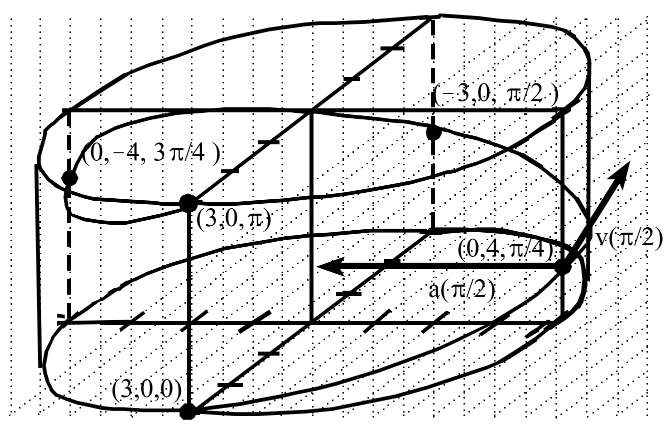
   $\mathbf{a}(t) = \mathbf{v}'(t) = -3\cos(t)\,\vec{\mathbf{i}} - 4\sin(t)\,\vec{\mathbf{j}} + 0\vec{\mathbf{k}} = -\,3\cos(t)\,\vec{\mathbf{i}} - 4\sin(t)\,\vec{\mathbf{j}}$

   At $t = \frac{\pi}{2}$, we have:

   $\mathbf{v}\left(\frac{\pi}{2}\right) = -3\sin\left(\frac{\pi}{2}\right)\vec{\mathbf{i}} + 4\cos\left(\frac{\pi}{2}\right)\vec{\mathbf{j}} + \frac{1}{2}\vec{\mathbf{k}} = -\,3\vec{\mathbf{i}} + \frac{1}{2}\vec{\mathbf{k}}$

   $\mathbf{a}\left(\frac{\pi}{2}\right) = -\,3\cos\left(\frac{\pi}{2}\right)\vec{\mathbf{i}} - 4\sin\left(\frac{\pi}{2}\right)\vec{\mathbf{j}} = -\,4\vec{\mathbf{j}}$

   The vectors are shown below.



3. Evaluate: $\int_{-2}^{3}\left(t\vec{\mathbf{i}} + t^3\vec{\mathbf{j}} + 4t\vec{\mathbf{k}}\right)dt =$

   The vectors is integrated "component-wise."

   $\int_{-2}^{3}\left(t\vec{\mathbf{i}} + t^3\vec{\mathbf{j}} + 4t\vec{\mathbf{k}}\right)dt = \left(\int_{-2}^{3} t\,dt\right)\vec{\mathbf{i}} + \left(\int_{-2}^{3} t^3\,dt\right)\vec{\mathbf{j}} + \left(\int_{-2}^{3} 4t\,dt\right)\vec{\mathbf{k}} =$

   $\left(\left[\frac{1}{2}t^2\right]_{-2}^{3}\right)\vec{\mathbf{i}} + \left(\left[\frac{1}{4}t^4\right]_{-2}^{3}\right)\vec{\mathbf{j}} + \left(\left[2t^2\right]_{-2}^{3}\right)\vec{\mathbf{k}} = \left(\frac{1}{2}(3)^2 - \frac{1}{2}(-2)^2\right)\vec{\mathbf{i}} + \left(\frac{1}{4}(3)^4 - \frac{1}{4}(-2)^4\right)\vec{\mathbf{j}} +$
   $\left(2(3)^2 - 2(-2)^2\right)\vec{\mathbf{k}} =$

   $\frac{5}{2}\vec{\mathbf{i}} + \frac{65}{4}\vec{\mathbf{j}} + 10\vec{\mathbf{k}}$

4. $a(t) = 4\vec{\mathbf{i}} + 3\vec{\mathbf{k}}$. Compute the velocity and position vectors, given the following information:

   (a) $v(0) = 4\vec{\mathbf{j}} + \vec{\mathbf{k}}$

   (b) $r(0) = 2\vec{\mathbf{i}}$

   $v(t) = \int a(t)\ dt = \int \left(4\vec{\mathbf{i}} + 3\vec{\mathbf{k}}\right)\ dt = 4t\vec{\mathbf{i}} + 3t\vec{\mathbf{k}} + \mathbf{C}$

   i.e., $v(t) = 4t\vec{\mathbf{i}} + 3t\vec{\mathbf{k}} + \mathbf{C}$

   From the information given, we have: $v(0) = 4\vec{\mathbf{j}} + \vec{\mathbf{k}}$

   $\Rightarrow v(0) = 4(0)\vec{\mathbf{i}} + 3(0)\vec{\mathbf{k}} + C = C = 4\vec{\mathbf{j}} + \vec{\mathbf{k}}$

   i.e., $C = 4\vec{\mathbf{j}} + \vec{\mathbf{k}}$

   $\Rightarrow v(t) = 4t\vec{\mathbf{i}} + 3t\vec{\mathbf{k}} + C = 4t\vec{\mathbf{i}} + 3t\vec{\mathbf{k}} + \left(4\vec{\mathbf{j}} + \vec{\mathbf{k}}\right) = 4(t+1)\vec{\mathbf{i}} + (3t+1)\vec{\mathbf{k}}$

   i.e., $v(t) = 4(t+1)\vec{\mathbf{i}} + (3t+1)\vec{\mathbf{k}}$

   To find the position function, realize that $r(t) = \int v(t)\ dt = \int \left(4(t+1)\vec{\mathbf{i}} + (3t+1)\vec{\mathbf{k}}\right) dt = (2t^2 + t)\vec{\mathbf{i}} + \left(\frac{3}{2}t^2 + t\right)\vec{\mathbf{k}} + C_2$

   i.e., $r(t) = (2t^2 + t)\vec{\mathbf{i}} + \left(\frac{3}{2}t^2 + t\right)\vec{\mathbf{k}} + C_2$

   From the information given, we have: $r(0) = 2\vec{\mathbf{i}}$

   $\Rightarrow r(0) = \left(2(0)^2 + (0)\right)\vec{\mathbf{i}} + \left(\frac{3}{2}(0)^2 + (0)\right)\vec{\mathbf{k}} + C_2 = C_2 = 2\vec{\mathbf{i}}$

   i.e., $C_2 = 2\vec{\mathbf{i}}$

   $\Rightarrow r(t) = (2t^2 + t)\vec{\mathbf{i}} + \left(\frac{3}{2}t^2 + t\right)\vec{\mathbf{k}} + C_2 = (2t^2 + t)\vec{\mathbf{i}} + \left(\frac{3}{2}t^2 + t\right)\vec{\mathbf{k}} + 2\vec{\mathbf{i}} = (2t^2 + t + 2)\vec{\mathbf{i}} + \left(\frac{3}{2}t^2 + t\right)\vec{\mathbf{k}}$

5. Compute the speed of the vector in problem 4 at the point $t = 4$ sec.

   Recall: speed $= ||v(t)||$

   At $t = 4$, speed $= ||v(4)|| = \left|\left|4((4)+1)\vec{\mathbf{i}} + (3(4)+1)\vec{\mathbf{k}}\right|\right| = \left|\left|20\vec{\mathbf{i}} + 13\vec{\mathbf{k}}\right|\right| = \sqrt{20^2 + 13^2} = \sqrt{569}$

6. Given the vector $r(t) = 2\sin(t)\vec{\mathbf{i}} + 2\cos(t)\vec{\mathbf{j}}$, compute the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and evaluate them at $t = \frac{\pi}{4}$.

   $\mathbf{T}(t) = \frac{r'(t)}{||r'(t)||} = \frac{2\cos(t)\vec{\mathbf{i}} - 2\sin(t)\vec{\mathbf{j}}}{\left|\left|2\cos(t)\vec{\mathbf{i}} - 2\sin(t)\vec{\mathbf{j}}\right|\right|} = \frac{2\cos(t)\vec{\mathbf{i}} - 2\sin(t)\vec{\mathbf{j}}}{\sqrt{4\cos^2(t) + 4\sin^2(t)}} = \frac{2\cos(t)}{\sqrt{4\cos^2(t) + 4\sin^2(t)}}\vec{\mathbf{i}} - \frac{2\sin(t)}{\sqrt{4\cos^2(t) + 4\sin^2(t)}}\vec{\mathbf{j}} = \frac{2\cos(t)}{\sqrt{4\left(\cos^2(t) + \sin^2(t)\right)}}\vec{\mathbf{i}} - \frac{2\sin(t)}{\sqrt{4\left(\cos^2(t) + \sin^2(t)\right)}}\vec{\mathbf{j}} = \frac{2\cos(t)}{2}\vec{\mathbf{i}} - \frac{2\sin(t)}{2}\vec{\mathbf{j}} = \cos(t)\vec{\mathbf{i}} - \sin(t)\vec{\mathbf{j}}$

   At $t = \frac{\pi}{4}$, we have $\mathbf{T}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)\vec{\mathbf{i}} - \sin\left(\frac{\pi}{4}\right)\vec{\mathbf{j}} = \frac{\sqrt{2}}{2}\vec{\mathbf{i}} - \frac{\sqrt{2}}{2}\vec{\mathbf{j}}$

   $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \frac{-\sin(t)\vec{\mathbf{i}} - \cos(t)\vec{\mathbf{j}}}{\left|\left|-\sin(t)\vec{\mathbf{i}} - \cos(t)\vec{\mathbf{j}}\right|\right|} = \frac{-\sin(t)\vec{\mathbf{i}} - \cos(t)\vec{\mathbf{j}}}{\sqrt{(-\sin(t))^2 + (-\cos(t))^2}} = \frac{-\sin(t)\vec{\mathbf{i}} - \cos(t)\vec{\mathbf{j}}}{\sqrt{\sin^2(t) + \cos^2(t)}} = -\sin(t)\vec{\mathbf{i}} - \cos(t)\vec{\mathbf{j}}$

   At $t = \frac{\pi}{4}$, we have $\mathbf{N}\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)\vec{\mathbf{i}} - \cos\left(\frac{\pi}{4}\right)\vec{\mathbf{j}} = -\frac{\sqrt{2}}{2}\vec{\mathbf{i}} - \frac{\sqrt{2}}{2}\vec{\mathbf{j}}$

7. For the vector in problem 6, evaluate $\mathbf{a_T}$, and $\mathbf{a_N}$ at $t = \frac{\pi}{4}$.

   First, we need to compute the acceleration, $a(t)$.

   $a(t) = v'(t) = r''(t) = -\sin(t)\,\vec{\mathbf{i}} - \cos(t)\,\vec{\mathbf{j}}$

   $\mathbf{a_T} = a \circ \mathbf{T} = \left(-\sin(t)\,\vec{\mathbf{i}} - \cos(t)\,\vec{\mathbf{j}}\right) \circ \left(\cos(t)\,\vec{\mathbf{i}} - \sin(t)\,\vec{\mathbf{j}}\right) = -\sin(t)\cos(t) + \cos(t)\sin(t) = 0$

   $\mathbf{a_N} = a \circ \mathbf{N} = \left(-\sin(t)\,\vec{\mathbf{i}} - \cos(t)\,\vec{\mathbf{j}}\right) \circ \left(-\sin(t)\,\vec{\mathbf{i}} - \cos(t)\,\vec{\mathbf{j}}\right) = \sin^2(t) + \cos^2(t) = 1$

8. Compute $\frac{\partial}{\partial x}\left[\sin(x^2 y) + x^3 - 5y^4\right]$

   When we compute the partial derivative with respect, we treat all variables except for x as though they were constants.

   $\frac{\partial}{\partial x}\left[\sin(x^2 y) + x^3 - 5y^4\right] = \underbrace{\cos\left(x^2 y\right) \cdot 2x}_{\text{chain rule}} + 3x^2 + 0 = \cos(x^2 y) \cdot 2x + 3x^2$

9. Given $f(x, y) = \sin(x - 2y)$, show that $f_{xy} = f_{yx}$

   $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left[f(x, y)\right]\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left[\sin(x - 2y)\right]\right) = \frac{\partial}{\partial y}\left(\cos(x - 2y)\right) = -\sin(x - 2y) \cdot (-2) = 2\sin(x - 2y)$

   $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left[f(x, y)\right]\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left[\sin(x - 2y)\right]\right) = \frac{\partial}{\partial x}\left(\cos(x - 2y)(-2)\right) = \frac{\partial}{\partial x}\left(-2\cos(x - 2y)\right) = -2\left(-\sin(x - 2y)\right) = 2\sin(x - 2y)$

   Hence, $f_{xy} = f_{yx}$

10. Find the relative maxes and mins of the function $f(x, y) = x^3 - y^3 - 3xy + 4$

    To find the critical numbers, we set the partial derivatives equal to 0.

    $\frac{\partial f}{\partial x} = 3x^2 - 3y = 0 \Rightarrow x^2 = y$

    $\frac{\partial f}{\partial y} = -3y^2 - 3x = 0 \Rightarrow y^2 = -x$

    Substituting $x^2 = y$ into $y^2 = -x$, we get $x^4 = -x \Rightarrow x^4 + x = 0 \Rightarrow x(x^3 + 1) = 0 \Rightarrow x = -1, 0$

    $x = -1 \Rightarrow y = 1 \Rightarrow (-1, 1)$ is a critical point.

    $x = 0 \Rightarrow y = 0 \Rightarrow 0, 0$ is a critical point.

    Now we need to test these. We need the second partial derivatives.

    $\frac{\partial^2 f}{\partial x^2} = 6x$

    $\frac{\partial^2 f}{\partial y^2} = -6y$

    $\frac{\partial^2 f}{\partial y \partial x} = -3$

    Recall: If $f_{xx} f_{yy} - \left(f_{xy}\right)^2 > 0$, then there is a relative extreme.

    If $f_{xx} f_{yy} - \left(f_{xy}\right)^2 < 0$, then there is NO relative extreme.

    At $(-1, 1)$, $\frac{\partial^2 f}{\partial x^2} = -6$, $\frac{\partial^2 f}{\partial y^2} = -6$, and $\frac{\partial^2 f}{\partial y \partial x} = -3$

    $\Rightarrow f_{xx} f_{yy} - \left(f_{xy}\right)^2 = (-6)(-6) - (-3)^2 = 27 > 0$, Therefore, we have an extreme. Since $\frac{\partial^2 f}{\partial x^2} = -6 < 0$, the surface is concave down, so we have a relative max at $(-1, 1)$

i.e., $(-1, 1, 5)$ is a local max.

At the point $(0, 0)$ $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, and $\frac{\partial^2 f}{\partial y \partial x} = -3$

$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = (0)(0) - (-3)^2 = -9 < 0$

Therefore, we have neither a rel max nor a rel min at $(0, 0, 4)$