MTH 2227 - Practice Test #4 - Solutions

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1. Compute $\frac{\partial}{\partial x}\left[\sin\left(x^2y\right)+x^3-5y^4\right]$

   When we compute the partial derivative with respect to x, we treat all variables except for x as though they were constants.

   $\frac{\partial}{\partial x}\left[\sin\left(x^2y\right)+x^3-5y^4\right]=\underbrace{\cos\left(x^2y\right)\cdot 2yx}_{\text{chain rule}}+3x^2+0=\cos\left(x^2y\right)\cdot 2yx+3x^2$

   $$\boxed{\frac{\partial}{\partial x}\left[\sin\left(x^2y\right)+x^3-5y^4\right]=2yx\cos\left(x^2y\right)+3x^2}$$

2. Given $f(x,y)=\sin(x-2y)$, show that $f_{xy}=f_{yx}$

   $f_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left[f(x,y)\right]\right)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left[\sin(x-2y)\right]\right)=\frac{\partial}{\partial y}\left(\cos(x-2y)\right)=-\sin(x-2y)\cdot(-2)=2\sin(x-2y)$

   $f_{yx}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left[f(x,y)\right]\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left[\sin(x-2y)\right]\right)=\frac{\partial}{\partial x}\left(\cos(x-2y)(-2)\right)=\frac{\partial}{\partial x}\left(-2\cos(x-2y)\right)$

   $=-2\left(-\sin(x-2y)\right)=2\sin(x-2y)$

   $$\boxed{f_{xy}=2\sin(x-2y)=f_{yx}}$$

3. Convert from Cylindrical to Rectangular Coordinates: $(r,\theta,z)=\left(6\sqrt{2},\frac{\pi}{4},8\right)$

   $x=r\cos(\theta)=\left(6\sqrt{2}\right)\cos\left(\frac{\pi}{4}\right)=\left(6\sqrt{2}\right)\left(\frac{\sqrt{2}}{2}\right)=3$

   $y=r\sin(\theta)=\left(6\sqrt{2}\right)\sin\left(\frac{\pi}{4}\right)=\left(6\sqrt{2}\right)\left(\frac{\sqrt{2}}{2}\right)=3$

   $z=8$

   $$\boxed{(x,y,z)=(3,3,8)}$$

4. Convert from Spherical to Rectangular Coordinates: $(\rho,\theta,\phi)=\left(8,\frac{\pi}{6},\frac{2\pi}{3}\right)$

   $x=\rho\sin(\phi)\cos(\theta)=8\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{6}\right)=8\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)=6$

   i.e., $x=6$

   $y=\rho\sin(\phi)\sin(\theta)=8\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)=8\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)=2\sqrt{3}$

   i.e., $y=2\sqrt{3}$

   $z=\rho\cos(\theta)=8\cos\left(\frac{\pi}{6}\right)=8\left(\frac{\sqrt{3}}{2}\right)=4\sqrt{3}$

   $$\boxed{(x,y,z)=\left(6,2\sqrt{3},4\sqrt{3}\right)}$$

5. Convert from Rectangular to Cylindrical and Spherical Coordinates: $(x, y, z) = \left(2, 2\sqrt{3}, 3\right)$ (Do not attempt to convert ϕ into increments of π)

Cylindrical

$$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + \left(2\sqrt{3}\right)^2} = \sqrt{16} = 4$$

i.e., $r = 4$

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{2\sqrt{3}}{2}\right) = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$$

i.e., $\theta = \frac{\pi}{3}$

$z = 3$

So $(r, \theta, z) = \left(4, \frac{\pi}{3}, 3\right)$

Spherical

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(2)^2 + \left(2\sqrt{3}\right)^2 + (3)^2} = 5$$

i.e., $\rho = 5$

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{2\sqrt{3}}{2}\right) = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$$

i.e., $\theta = \frac{\pi}{3}$

$$\phi = \arctan\left(\frac{r}{z}\right) = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \arctan\left(\frac{\sqrt{(2)^2+\left(2\sqrt{3}\right)^2}}{3}\right) = \arctan\left(\frac{4}{3}\right)$$

Alternatively:

$$\phi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \arccos\left(\frac{3}{\sqrt{(2)^2+\left(2\sqrt{3}\right)^2+(3)^2}}\right) = \arccos\left(\frac{3}{5}\right)$$

i.e., $(\rho, \theta, \phi) = \left(5, \frac{\pi}{3}, \arctan\left(\frac{4}{3}\right)\right)$

Alternatively:

i.e., $(\rho, \theta, \phi) = \left(5, \frac{\pi}{3}, \arccos\left(\frac{3}{5}\right)\right)$

$(r, \theta, z) = \left(4, \frac{\pi}{3}, 3\right)$

$(\rho, \theta, \phi) = \left(5, \frac{\pi}{3}, \arctan\left(\frac{4}{3}\right)\right) = \left(5, \frac{\pi}{3}, \arccos\left(\frac{3}{5}\right)\right)$

6. Find the relative maxes and mins of the function $f(x,y) = x^3 - y^3 - 3xy + 4$

To find the critical numbers, we set the partial derivatives equal to 0.

$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0 \quad \text{(Eq. 1)}$

$\Rightarrow y = x^2 \quad \text{(Eq. 1a)}$

$\frac{\partial f}{\partial y} = -3y^2 - 3x = 0 \quad \text{(Eq. 2)}$

$\Rightarrow y^2 = -x \quad \text{(Eq. 2a)}$

Substituting $y = x^2$ into Eq. 2, we have:

$(x^2)^2 = -x$
$\Rightarrow x^4 = -x$
$\Rightarrow x^4 + x = 0$
$\Rightarrow x(x^3 + 1) = 0 \Rightarrow x = -1, 0$

Plugging $x = -1$ into Eq. 1a, we get $y = 1 \Rightarrow (-1, 1)$ is a critical point.

Plugging $x = 0$ into Eq. 1a, we get $y = 0 \Rightarrow (0, 0)$ is a critical point.

Now we need to test these. We need the second partial derivatives.

$\frac{\partial^2 f}{\partial x^2} = 6x$

$\frac{\partial^2 f}{\partial y^2} = -6y$

$\frac{\partial^2 f}{\partial y \partial x} = -3$

Recall: If the discriminant $f_{xx} f_{yy} - (f_{xy})^2 > 0$, then there is a relative extreme.

If $f_{xx} f_{yy} - (f_{xy})^2 < 0$, then there is NO relative extreme.

At $(-1, 1)$, $\frac{\partial^2 f}{\partial x^2} = -6$, $\frac{\partial^2 f}{\partial y^2} = -6$, and $\frac{\partial^2 f}{\partial y \partial x} = -3$

$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = (-6)(-6) - (-3)^2 = 27 > 0$, Therefore, we have an extreme. Since $\frac{\partial^2 f}{\partial x^2} = -6 < 0$, the surface is concave down, so we have a relative max at $(-1, 1)$

i.e., $(-1, 1, 5)$ is a relative max.

At the point $(0, 0)$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, and $\frac{\partial^2 f}{\partial y \partial x} = -3$

$\Rightarrow f_{xx} f_{yy} - (f_{xy})^2 = (0)(0) - (-3)^2 = -9 < 0$

Therefore, we have neither a rel max nor a rel min at $(0, 0, 4)$

The point $(x, y, z) = (-1, 1, 5)$ is a relative max.

The point $(x, y, z) = (0, 0, 4)$ is neither a relative max nor a relative min

7. Convert the following equations from Rectangular Coordinates to Cylindrical Coordinates

(a) $x^2 + y^2 + z^2 = 4$

Recall: $x^2 + y^2 = r^2$

Thus, our equation becomes $r^2 + z^2 = 4$

$$\boxed{r^2 + z^2 = 4}$$

(b) $3x + y - 4z = 12$

Recall: $x = r\cos(\theta)$ and $y = r\sin(\theta)$

Thus our equation $\underbrace{3x}_{3r\cos(\theta)} + \underbrace{y}_{r\sin(\theta)} - 4z = 12$ becomes $3r\cos(\theta) + r\sin(\theta) - 4z = 12$

$$\boxed{3r\cos(\theta) + r\sin(\theta) - 4z = 12}$$

(c) $y^2 + z^2 = 9$

Recall: $y = r\sin(\theta)$

Thus our equation $\underbrace{y^2}_{(r\sin(\theta))^2} + z^2 = 9$ becomes $(r\sin(\theta))^2 + z^2 = 9$

$$\boxed{r^2\sin^2(\theta) + z^2 = 9}$$

(d) $6x = x^2 + y^2$

Recall: $x^2 + y^2 = r^2$

$x = r\cos(\theta)$

$y = r\sin(\theta)$

Thus our equation $\underbrace{6x}_{6r\cos(\theta)} = \underbrace{x^2 + y^2}_{r^2}$ becomes $6r\cos(\theta) = r^2$

$\Rightarrow 6\cos(\theta) = r$

$$\boxed{6\cos(\theta) = r}$$

(e) $y = xz$

Recall: $x = r\cos(\theta)$ and $y = r\sin(\theta)$

Thus our equation $\underbrace{y}_{r\sin(\theta)} = \underbrace{x}_{r\cos(\theta)} z$ becomes $r\sin(\theta) = r\cos(\theta)\, z$

$\Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = z$

$\Rightarrow \tan(\theta) = z$

$$\boxed{\tan(\theta) = z}$$

8. Convert the following equations from Rectangular Coordinates to Spherical Coordinates

(a) $x^2 + y^2 + z^2 = 4$

Recall: $x^2 + y^2 + z^2 = \rho^2$
Thus, our equation $\underbrace{x^2 + y^2 + z^2}_{\rho^2} = 4$ becomes $\rho^2 = 4$

$\Rightarrow \rho = 2$

$$\boxed{\rho = 2}$$

(b) $3x + y - 4z = 12$

Recall:

$$x = \rho\sin(\phi)\cos(\theta)$$
$$y = \rho\sin(\phi)\sin(\theta)$$
$$z = \rho\cos(\phi)$$

Thus, our equation $\underbrace{3x}_{3\rho\sin(\phi)\cos(\theta)} + \underbrace{y}_{\rho\sin(\phi)\sin(\theta)} - \underbrace{4z}_{4\rho\cos(\phi)} = 12$ becomes:

$3\rho\sin(\phi)\cos(\theta) + \rho\sin(\phi)\sin(\theta) - 4\rho\cos(\phi) = 12$

$$\boxed{\rho\sin(\phi)\cos(\theta) + \rho\sin(\phi)\sin(\theta) - 4\rho\cos(\phi) = 12}$$

(c) $y^2 + z^2 = 9$

Recall: $x = \rho \sin(\phi) \cos(\theta)$

$y = \rho \sin(\phi) \sin(\theta)$

$z = \rho \cos(\phi)$

Thus, our equation $\underbrace{y^2}_{(\rho \sin(\phi) \sin(\theta))^2} + \underbrace{z^2}_{(\rho \cos(\phi))^2} = 9$ becomes:

$(\rho \sin(\phi) \sin(\theta))^2 + (\rho \cos(\phi))^2 = 9$

$\Rightarrow \rho^2 \sin^2(\phi) \sin^2(\theta) + \rho^2 \cos^2(\phi) = 9$

$\Rightarrow \rho^2 \left(\sin^2(\phi) \sin^2(\theta) + \cos^2(\phi)\right) = 9$

$$\boxed{\rho^2 \left(\sin^2(\phi) \sin^2(\theta) + \cos^2(\phi)\right) = 9}$$

(d) $6x = x^2 + y^2$

Recall: $x = \rho \sin(\phi) \cos(\theta)$

$y = \rho \sin(\phi) \sin(\theta)$

$z = \rho \cos(\phi)$

Thus, our equation $\underbrace{6x}_{\rho \sin(\phi) \cos(\theta)} = \underbrace{x^2}_{(\rho \sin(\phi) \cos(\theta))^2} + \underbrace{y^2}_{(\rho \sin(\phi) \sin(\theta))^2}$ becomes:

$6\rho \sin(\phi) \cos(\theta) = (\rho \sin(\phi) \cos(\theta))^2 + (\rho \sin(\phi) \sin(\theta))^2$

$\Rightarrow 6\rho \sin(\phi) \cos(\theta) = \rho^2 \sin^2(\phi) \cos^2(\theta) + \rho^2 \sin^2(\phi) \sin^2(\theta)$

$\Rightarrow 6\rho \sin(\phi) \cos(\theta) = \rho^2 \sin^2(\phi) \left(\cos^2(\theta) + \sin^2(\theta)\right)$

$\Rightarrow 6\rho \sin(\phi) \cos(\theta) = \rho^2 \sin^2(\phi)$

$\Rightarrow 6 \cos(\theta) = \rho \sin(\phi)$

$$\boxed{\rho \sin(\phi) = 6 \cos(\theta)}$$

(e) $y = xz$

Recall: $x = \rho \sin(\phi) \cos(\theta)$

$y = \rho \sin(\phi) \sin(\theta)$

$z = \rho \cos(\phi)$

Thus, our equation $\underbrace{y}_{\rho \sin(\phi) \sin(\theta)} = \underbrace{x}_{\rho \sin(\phi) \cos(\theta)} \cdot \underbrace{z}_{\rho \cos(\phi)}$ becomes:

$\rho \sin(\phi) \sin(\theta) = (\rho \sin(\phi) \cos(\theta)) (\rho \cos(\phi))$

$\Rightarrow \sin(\theta) = \cos(\theta) \rho \cos(\phi)$

$\Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = \rho \cos(\phi)$

$\Rightarrow \tan(\theta) = \rho \cos(\phi)$

$$\boxed{\rho \cos(\phi) = \tan(\theta)}$$

9. Convert to Cylindrical Coordinates and Integrate: $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{3+\sqrt{9-x^2-y^2}} 1 dzdydx$

$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{3+\sqrt{9-x^2-y^2}} 1 dzdydx = \int_{0}^{2\pi} \int_{0}^{3} \int_{0}^{3+\sqrt{9-r^2}} r dzdrd\theta = 45\pi$

10. Convert to Spherical Coordinates and Integrate: $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{9-x^2-y^2}} 1 dzdydx$

$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{9-x^2-y^2}} 1 dzdydx = \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{4}} \int_{0}^{3} \rho^2 \sin(\phi) \, d\rho d\phi d\theta = 9\pi \left(2 - \sqrt{2}\right)$