Laplace Transforms Homework #2

SPRING 2001

Pat Rossi Name ____________________

Instructions. For problems 1 - 10, find the Laplace Transform of each function.

Remark: *We will make copious use of two fundamental properties of Laplace Transforms:*

1) $\pounds\left[f(t) \pm g(t)\right] = \pounds\left[f(t)\right] \pm \pounds\left[g(t)\right]$

2) $\pounds\left[c \cdot f(t)\right] = c \cdot \pounds\left[f(t)\right]$

1. $5 - 8t^3$

   $\pounds\left[5 - 8t^3\right] = \pounds\left[5\right] - \pounds\left[8t^3\right] = \pounds\left[5\right] - 8\pounds\left[t^3\right] = \frac{5}{s} - 8\frac{3!}{s^4} = \frac{5}{s} - \frac{48}{s^4}$

   (Using Formulas #1 and #4 on the Laplace Transforms Table)

   i.e. $\pounds\left[5 - 8t^3\right] = \frac{5}{s} - \frac{48}{s^4}$

2. $\frac{1}{8}\cos\left(\frac{3}{8}t\right)$

   $\pounds\left[\frac{1}{8}\cos\left(\frac{3}{8}t\right)\right] = \frac{1}{8}\pounds\left[\cos\left(\frac{3}{8}t\right)\right] = \frac{1}{8}\frac{s}{s^2+\left(\frac{3}{8}\right)^2} = \frac{s}{8s^2+\frac{9}{8}}$

   (Using Formula #9 on the Laplace Transforms Table)

   i.e. $\pounds\left[\frac{1}{8}\cos\left(\frac{3}{8}t\right)\right] = \frac{s}{8s^2+\frac{9}{8}} = \frac{8s}{64s^2+9}$

3. $e^{3t}\cos(2t) - e^t\sinh(5t)$

4. $\cos(t) - \sin(t)$

   $\pounds\left[\cos(t) - \sin(t)\right] = \pounds\left[\cos(t)\right] - \pounds\left[\sin(t)\right] = \frac{s}{s^2+1} - \frac{1}{s^2+1} = \frac{s-1}{s^2+1}$

   (Using Formulas #9,10 on the Laplace Transforms Table)

   i.e. $\pounds\left[\cos(t) - \sin(t)\right] = \frac{s-1}{s^2+1}$

5. $t^7 - t^4 + 5t^2$

$\pounds\left[t^7 - t^4 + 5t^2\right] = \pounds\left[t^7\right] - \pounds\left[t^4\right] + \pounds\left[5t^2\right] = \pounds\left[t^7\right] - \pounds\left[t^4\right] + 5\pounds\left[t^2\right] = \frac{7!}{s^{7+1}} - \frac{4!}{s^{4+1}} + 5\frac{2!}{s^{2+1}}$

(Using Formulas #3, 5 on the Laplace Transforms Table)

i.e. $\pounds\left[t^7 - t^4 + 5t^2\right] = \frac{7!}{s^8} - \frac{4!}{s^5} + \frac{10}{s^3}$

6. $t\sinh(t)$

7. $\frac{d}{dt}\left[te^{5t}\right]$

$\pounds\left(\frac{d}{dt}\left[te^{5t}\right]\right) =???$

By Formula #19, $\pounds\left[f'(t)\right] = sF(s) - f(0)$ (Here, $F(s) = \pounds\left[f(t)\right]$)

Thus, Formula #19 could be rewritten as: $\pounds\left[f'(t)\right] = s\pounds\left[f(t)\right] - f(0)$

$f(t) = te^{5t}$

$F(s) = \pounds\left[f(t)\right] = \pounds\left[te^{5t}\right] = \frac{1}{(s-5)^2}$

(Using Formula #14 on the Laplace Transforms Table)

Thus, $\pounds\left(\frac{d}{dt}\left[te^{5t}\right]\right) = \pounds\left[f'(t)\right] = sF(s) - f(0) = s\frac{1}{(s-5)^2} - (0)\,e^{5(0)} = \frac{s}{(s-5)^2}$

i.e. $\pounds\left(\frac{d}{dt}\left[te^{5t}\right]\right) = \frac{s}{(s-5)^2}$

8. $\frac{d^2}{dt^2}\left[\cos(t) + te^t\right]$

$\pounds\left(\frac{d^2}{dt^2}\left[\cos(t) + te^t\right]\right) = ???$

By Formula #20, $\pounds\left[f''(t)\right] = s^2F(s) - sf(0) - f'(0)$ (Here, $F(s) = \pounds\left[f(t)\right]$)

Thus, Formula #20 could be rewritten as: $\pounds\left[f''(t)\right] = s^2\pounds\left[f(t)\right] - sf(0) - f'(0)$

$f(t) = \cos(t) + te^t$

$f'(t) = -\sin(t) + te^t + e^t$

$F(s) = \pounds\left[f(t)\right] = \pounds\left[\cos(t) + te^t\right] = \pounds\left[\cos(t)\right] + \pounds\left[te^t\right] = \frac{s}{s^2+1} + \frac{1}{(s-1)^2}$

(Using Formula #9, 14 on the Laplace Transforms Table)

Thus, $\pounds\left(\frac{d^2}{dt^2}\left[\cos(t) + te^t\right]\right) = \pounds\left[f''(t)\right] = s^2F(s) - sf(0) - f'(0)$

$= s^2\left(\frac{s}{s^2+1} + \frac{1}{(s-1)^2}\right) - s\left(\cos(0) + (0)e^{(0)}\right) - \left(-\sin(0) + (0)e^{(0)} + e^{(0)}\right)$

$= \frac{s^3}{s^2+1} + \frac{s^2}{(s-1)^2} - s - 1$

$$\boxed{\text{i.e. } \pounds\left(\frac{d^2}{dt^2}\left[\cos(t) + te^t\right]\right) = \frac{s^3}{s^2+1} + \frac{s^2}{(s-1)^2} - s - 1}$$

9. $\int_0^t \cosh(z)\cos(t-z)\,dz$

10. $\int_0^t e^z\cos(2z)\,dz$

Further Instructions For problems 11 - 20, find the Inverse Laplace Transform of each function.

Remark: *We will make copious use of two fundamental properties of Laplace Transform Inverses:*

1) $\pounds^{-1}\left[F(s) \pm G(s)\right] = \pounds^{-1}\left[F(s)\right] \pm \pounds^{-1}\left[G(s)\right]$

2) $\pounds\left[c \cdot F(s)\right] = c \cdot \pounds\left[F(s)\right]$

11. $\frac{2}{s^2+k^2}$

$\pounds^{-1}\left[\frac{2}{s^2+k^2}\right] = 2\pounds^{-1}\left[\frac{1}{s^2+k^2}\right] = ???$

I would like to use Formula #6 on *A Table of Laplace Transform Inverses:* $\pounds^{-1}\left[\frac{k}{s^2+k^2}\right] = \sin(kt)$

In order to use this formula, I need to get the constant "k" in the numerator.

$\pounds^{-1}\left[\frac{2}{s^2+k^2}\right] = 2\pounds^{-1}\left[\frac{1}{s^2+k^2}\right] = 2\pounds^{-1}\left[\frac{1}{k}\frac{k}{s^2+k^2}\right] = 2\pounds^{-1}\left[\frac{1}{k}\frac{k}{s^2+k^2}\right] = \frac{2}{k}\pounds^{-1}\left[\frac{k}{s^2+k^2}\right] = \frac{2}{k}\sin(kt)$

i.e., $\pounds^{-1}\left[\frac{2}{s^2+k^2}\right] = \frac{2}{k}\sin(kt)$

12. $\frac{n!}{(s-k)^{n+1}}$; $n = 1, 2, 3, \ldots$

i.e., $\pounds^{-1}\left[\frac{n!}{(s-k)^{n+1}}\right] = t^n e^{kt}$

(Using Formula #5 on *A Table of Laplace Transform Inverses)*

13. $\frac{s}{s^2-k^2}$

i.e., $\pounds^{-1}\left[\frac{s}{s^2-k^2}\right] = \cosh(kt)$

(Using Formula #10 on *A Table of Laplace Transform Inverses)*

14. $\frac{2}{(s^2+1)^2}$

This doesn't really fit any of our forms:

We have: $\frac{2ks}{(s^2+k^2)^2}$ (Formula #13) and $\frac{s^2-k^2}{(s^2+k^2)^2}$ (Formula #14), but we don't have a formula with just a plain old **constant** over $(k^2+s^2)^2$.

If we are to use either (or both) of these forms, $k=1$.

So our forms will be: $\mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = t\sin(t)$ and $\mathcal{L}^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = t\cos(kt)$

Q: Can we do anything with these?

Observe: $\frac{s^2-1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$ (Using Partial Fraction Decomposition.)

$\Rightarrow \frac{2}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}$

Thus, $\mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = \sin(t) - t\cos(t)$

(Using Formulas #6, 14 on *A Table of Laplace Transform Inverses)*

i.e., $\mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right] = \sin(t) - t\cos(t)$

15. $\frac{s^2+3s+36}{s(s^2+13s+36)}$

$\mathcal{L}^{-1}\left[\frac{s^2+3s+36}{s(s^2+13s+36)}\right] = ???$

This doesn't even come remotely close to fitting any form given on *A Table of Laplace Transform Inverses* - We must re-express $F(s)$, using Partial Fraction Decomposition

$\frac{s^2+3s+36}{s(s^2+13s+36)} = \frac{s^2+3s+36}{s(s+9)(s+4)} = \frac{c_1}{s} + \frac{c_2}{(s+9)} + \frac{c_3}{(s+4)}$

$\Rightarrow s^2 + 3s + 36 = c_1(s+9)(s+4) + c_2 s(s+4) + c_3 s(s+9)$

$\boxed{\text{Let } s = 0}$

$\Rightarrow 36 = c_1(9)(4) = 36c_1$

$\boxed{\Rightarrow c_1 = 1}$

$\boxed{\text{Let } s = -9}$

$\Rightarrow (-9)^2 + 3(-9) + 36 = c_2(-9)((-9)+4)$

i.e. $90 = 45c_2$

$\boxed{\Rightarrow c_2 = 2}$

$\boxed{\text{Let } s = -4}$

$\Rightarrow (-4)^2 + 3(-4) + 36 = c_3(-4)((-4)+9)$

i.e. $40 = -20c_3$

$\boxed{\Rightarrow c_3 = -2}$

$$\mathcal{L}^{-1}\left[\frac{s^2+3s+36}{s(s^2+13s+36)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{2}{(s+9)} - \frac{2}{(s+4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{(s+9)}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+4)}\right]$$

$$= 1 + 2e^{-9t} - 2e^{-4t}$$

(Using Formulas #1, 4 on *A Table of Laplace Transform Inverses)*

$$\boxed{\text{i.e., } \mathcal{L}^{-1}\left[\frac{s^2+3s+36}{s(s^2+13s+36)}\right] = 1 + 2e^{-9t} - 2e^{-4t}}$$

16. $\frac{s^2+4s+36}{(s^2-4)^2}$

Using Partial Fraction Decomposition, we have:

$\frac{s^2+4s+36}{(s^2-4)^2} = \frac{1}{s+2} - \frac{1}{s-2} + \frac{3}{(s-2)^2} + \frac{2}{(s+2)^2}$

Hence, $\mathcal{L}^{-1}\left[\frac{s^2+4s+36}{(s^2-4)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2} - \frac{1}{s-2} + \frac{2}{(s+2)^2} + \frac{3}{(s-2)^2}\right]$

$= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right]$

$= e^{-2t} - e^{2t} + 2te^{-2t} + 3te^{2t}$

(Using Formulas #4, 5 on *A Table of Laplace Transform Inverses)*

i.e., $\mathcal{L}^{-1}\left[\frac{s^2+4s+36}{(s^2-4)^2}\right] = e^{-2t} - e^{2t} + 2te^{-2t} + 3te^{2t}$

17. $\frac{s^2+2s+53}{(s+2)(s^2+49)}$

Using Partial Fraction Decomposition, we have:

$\frac{s^2+2s+53}{(s+2)(s^2+49)} = \frac{1}{s+2} + \frac{2}{s^2+49}$

Hence, $\mathcal{L}^{-1}\left[\frac{s^2+2s+53}{(s+2)(s^2+49)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2} + \frac{2}{s^2+49}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s^2+49}\right]$

I want $\mathcal{L}^{-1}\left[\frac{1}{s^2+49}\right]$ to fit formula #6 on *A Table of Laplace Transform Inverses:* $\mathcal{L}^{-1}\left[\frac{k}{s^2+k^2}\right] = \sin(kt)$

So, I will multiply the numerator by $k = 7$, and divide by $k = 7$.

$\mathcal{L}^{-1}\left[\frac{s^2+2s+53}{(s+2)(s^2+49)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2} + \frac{2}{s^2+49}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s^2+49}\right]$

$= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{7}\frac{7}{s^2+49}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{2}{7}\mathcal{L}^{-1}\left[\frac{7}{s^2+49}\right]$

$= e^{-2t} + \frac{2}{7}\sin(7t)$

(Using Formulas #4, 6 on *A Table of Laplace Transform Inverses)*

i.e., $\mathcal{L}^{-1}\left[\frac{s^2+2s+53}{(s+2)(s^2+49)}\right] = e^{-2t} + \frac{2}{7}\sin(7t)$

18. $\frac{s^2+3s-18}{s(s^2-6s+9)}$

Using Partial Fraction Decomposition, we have:

$\frac{s^2+3s-18}{s(s^2-6s+9)} = \frac{(s+6)(s-3)}{s(s-3)^2} = \frac{(s+6)}{s(s-3)} = \frac{3}{s-3} - \frac{2}{s}$

Hence, $\mathcal{L}^{-1}\left[\frac{s^2+3s-18}{s(s^2-6s+9)}\right] = \mathcal{L}^{-1}\left[\frac{3}{s-3} - \frac{2}{s}\right] = 3\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s}\right]$

$= 3e^{3t} - 2$

(Using Formulas #1, 4 on *A Table of Laplace Transform Inverses)*

i.e., $\mathcal{L}^{-1}\left[\frac{s^2+3s-18}{s(s^2-6s+9)}\right] = 3e^{3t} - 2$

19. $\frac{s^2-s+1}{s^3(s+1)}$

20. $\frac{2s-3}{(s+1)^2+16}$