Differential Equations Practice Test #1 - Solutions

Spring 2004

Pat Rossi Name ______________________________

Solutions

1. Classify the following according to **order** and **linearity.**

   (a) $y''' - 2y'' - 5y' + 6y = 0$

   Order 3 (because y''' is the highest order derivative) and **linear** (because all derivatives are raised to the first power)

   (b) $(y')^3 = y$

   Order 1 (because y' is the highest order derivative) and **non-linear** (because y' is raised to a power other than 1.)

   (c) $\frac{d^2s}{dt^2} = -9s$

   Order 2 (because $\frac{d^2s}{dt^2}$ is the highest order derivative) and **linear** (because all derivatives are raised to the first power)

   (d) $y'' - 3y' - 10y = 6e^x$

   Order 2 (because y'' is the highest order derivative) and **linear** (because all derivatives are raised to the first power)

2. Solve: $\frac{dy}{dx} = -\frac{x}{y}$; $y = 2$ when $x = 1$

   By Separation of Variables, we have:

   $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow ydy = -xdx \Rightarrow \int ydy = -\int xdx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + C \Rightarrow x^2 + y^2 = C_1$

   Now, we'll use the initial conditions to find C_1.

   $y = 2$ when $x = 1 \Rightarrow (1)^2 + (2)^2 = C_1 \Rightarrow 5 = C_1$

   $x^2 + y^2 = 5$ is the particular solution.

3. Show that the function $y = c_1e^x + c_2e^{-x} - 4x$ is a solution of the differential equation $y'' - y = 4x$. Given the initial conditions, $y(0) = 2$ and $y'(0) = 0$, obtain a particular solution.

Observe:

$y' = c_1e^x - c_2e^{-x} - 4$

$y'' = c_1e^x + c_2e^{-x}$

Plugging the expressions for y, y', and y'' into the equation, $y'' - y = 4x$, we have:

$$\underbrace{\left(c_1e^x + c_2e^{-x}\right)}_{y''} - \underbrace{\left(c_1e^x + c_2e^{-x} - 4x\right)}_{y} = 4x$$

Hence, $y = c_1e^x + c_2e^{-x} - 4x$ is a solution of the differential equation $y'' - y = 4x$.

To find the particular solution, given the initial conditions, $y(0) = 2$ and $y'(0) = 0$, observe:

$y(0) = 2 \Rightarrow 2 = c_1e^0 + c_2e^{-0} - 4(0) \Rightarrow c_1 + c_2 = 2$

Also:

$y'(0) = 0 \Rightarrow 0 = c_1e^0 - c_2e^{-0} - 4 \Rightarrow c_1 - c_2 - 4 = 0 \Rightarrow c_1 - c_2 = 4$

So we have two equations in two unknowns:

$$\begin{array}{ccccc} c_1 & + & c_2 & = & 2 \\ c_1 & - & c_2 & = & 4 \end{array}$$

Solving this system, we have:

$$c_1 = 3 \text{ and } c_2 = -1$$

Our particular solution is $y = 3e^x - e^{-x} - 4x$

4. Solve: $y' = 8xy + 3y \quad y(-1) = 1$ (Assume $y > 0$)

Method #1 (Separation of Variables)

$y' = 8xy + 3y \Rightarrow \frac{dy}{dx} = 8xy + 3y \Rightarrow \frac{dy}{dx} = (8x+3)\,y \Rightarrow \frac{1}{y}dy = (8x+3)\,dx$

Integrate!

$\int \frac{1}{y}dy = \int (8x+3)\,dx \Rightarrow \ln y = 4x^2 + 3x + C$

To find the particular solution, consider the initial conditions, $y(-1) = 1$.

$\Rightarrow \ln 1 = 4(-1)^2 + 3(-1) + C \Rightarrow C = -1$

Thus, we have: $\ln y = 4x^2 + 3x - 1 \Rightarrow e^{\ln y} = e^{4x^2+3x-1} \Rightarrow y = e^{4x^2+3x-1}$ (particular solution).

Method #2 (Linear First Order $y' + P(x)\,y = Q(x)$)

Rewrite: $y' - (8x+3)\,y = 0$

Rewrite as $y' + \underbrace{[-(8x+3)]}_{P(x)}y = \underbrace{0}_{Q(x)}$

(a) 1. Compute the integrating factor,

$e^{\int P(x)dx} = e^{\int -(8x+3)dx} = e^{-4x^2-3x}$

2. Multiply both sides by the integrating factor.

$\Rightarrow y'e^{-4x^2-3x} - e^{-4x^2-3x}(8x+3)\,y = 0$

3. Recognize that the Left Hand Side is the derivative of a product (Specifically, it's the derivative of y times the integrating factor)

$\Rightarrow \frac{d}{dx}\left[ye^{-4x^2-3x}\right] = 0$

4. Integrate:

$\Rightarrow \int \frac{d}{dx}\left[ye^{-4x^2-3x}\right] dx = \int 0dx \Rightarrow ye^{-4x^2-3x} = C$

5. Divide both sides by the integrating factor

$\Rightarrow y = Ce^{4x^2+3x}$

To find the particular solution, consider the initial conditions, $y(-1) = 1$.

$\Rightarrow 1 = Ce^{4(-1)^2+3(-1)} \Rightarrow 1 = Ce \Rightarrow C = e^{-1}$

Hence, $y = e^{-1}e^{4x^2+3x} = e^{4x^2+3x-1}$

i.e., $y = e^{4x^2+3x-1}$ is out particular solution.

5. Solve: $xdy = (2y + 3x)\,dx$

Method #1 (Linear First Order)

Rewrite: $\frac{dy}{dx} = \frac{2}{x}y + 3 \quad \Rightarrow \frac{dy}{dx} - \frac{2}{x}y = 3 \quad \Rightarrow \frac{dy}{dx} + \underbrace{\left[-\frac{2}{x}\right]}_{P(x)} y = \underbrace{3}_{Q(x)}$

(a) 1. Compute the integrating factor:

$$e^{\int P(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = e^{\ln\left|x^{-2}\right|} = e^{\ln\left(x^{-2}\right)} = x^{-2}$$

2. Multiply both sides by the integrating factor:

$$\Rightarrow \frac{dy}{dx}x^{-2} - \frac{2}{x}x^{-2}y = 3x^{-2} \Rightarrow \frac{dy}{dx}x^{-2} - 2x^{-3}y = 3x^{-2}$$

3. Recognize that the left hand side is the derivative of a product. Specifically, the left hand side is the derivative of y times the integrating factor.

$$\Rightarrow \frac{d}{dx}\left[yx^{-2}\right] = 3x^{-2}$$

4. Integrate:

$$\Rightarrow \int \frac{d}{dx}\left[yx^{-2}\right]dx = \int 3x^{-2}dx \Rightarrow yx^{-2} = -3x^{-1} + C$$

5. Divide both sides by the integrating factor:

$$\Rightarrow y = -3x + Cx^2$$

Method #2 (Substitution)

Let $v = \frac{y}{x}$. Then $y = vx \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v$

Rewriting our equation, $xdy = (2y + 3x)\,dx$ as $\frac{dy}{dx} = \frac{2}{x}y + 3$, we can substitute our expressions involving v and $\frac{dv}{dx}$.

$$\Rightarrow \underbrace{\frac{dv}{dx}x + v}_{\frac{dy}{dx}} = \frac{2}{x}\underbrace{vx}_{y} + 3 \Rightarrow \frac{dv}{dx}x + v = 2v + 3$$

Separate:

$$\Rightarrow \frac{1}{v+3}dv = \frac{1}{x}dx$$

Integrate:

$$\Rightarrow \int \frac{1}{v+3}dv = \int \frac{1}{x}dx \Rightarrow \ln(v+3) = \ln(x) + C \Rightarrow e^{\ln(v+3)} = e^{\ln(x)+C}$$

$$\Rightarrow v + 3 = e^{\ln(x)}e^{C} \Rightarrow v + 3 = C_1 e^{\ln(x)} \Rightarrow v + 3 = C_1 x$$

Re-express in terms of y :

$$\Rightarrow \frac{y}{x} + 3 = C_1 x \Rightarrow y + 3x = C_1 x^2 \Rightarrow y = -3x + C_1 x^2$$

6. Solve: $I' + 3I = e^{-2t}; \quad I(0) = 5$

Observe: This fits the form:

$$I' + \underbrace{3}_{P(t)} I = \underbrace{e^{-2t}}_{Q(t)}$$

(a) 1. Compute the integrating factor:

$e^{\int P(t)dt} = e^{\int 3dt} = e^{3t}$

2. Multiply both sides by the integrating factor:

$\Rightarrow I'e^{3t} + e^{3t}3I = e^{3t}e^{-2t} \Rightarrow I'e^{3t} + e^{3t}3I = e^{t}$

3. Recognize that the left hand side is the derivative of a product. Specifically, the left hand side is the derivative of I times the integrating factor.

$\Rightarrow \frac{d}{dt}\left[Ie^{3t}\right] = e^{t}$

4. Integrate:

$\Rightarrow \int \frac{d}{dt}\left[Ie^{3t}\right] dt = \int e^{t} dt \Rightarrow Ie^{3t} = e^{t} + C$

5. Divide by the integrating factor:

$\Rightarrow I = e^{-2t} + Ce^{-3t}$

To find the particular solution, use the initial conditions, $I(0) = 5$.

$5 = e^{-2(0)} + Ce^{-3(0)} \Rightarrow 5 = 1 + C \Rightarrow C = 4$

Hence, our particular solution is: $I = e^{-2t} + 4e^{-3t}$

7. Solve: $\frac{dI}{dt}+\frac{10I}{2t+5}=10; \quad I(0)=0$ (Assume that $t\geq 0$)

This fits the form: $\frac{dI}{dt}+\underbrace{\frac{10}{2t+5}}_{P(t)} I = \underbrace{10}_{Q(t)}$

(a) 1. Compute the integrating factor

$$\Rightarrow e^{\int P(t)dt}=e^{\int \frac{10}{2t+5}dt}=e^{5\ln(2t+5)}=e^{\ln\left[(2t+5)^5\right]}=(2t+5)^5$$

2. Multiply both sides by the integrating factor

$$\Rightarrow \frac{dI}{dt}(2t+5)^5+(2t+5)^5\frac{10}{2t+5}I=10(2t+5)^5$$
$$\Rightarrow \frac{dI}{dt}(2t+5)^5+10(2t+5)^4 I=10(2t+5)^5$$

3. Recognize that the left hand side is the derivative of a product. Specifically, the left hand side is the derivative of I times the integrating factor.

$$\Rightarrow \frac{d}{dt}\left[I(2t+5)^5\right]=10(2t+5)^5$$

4. Integrate:

$$\int \frac{d}{dt}\left[I(2t+5)^5\right]dt=\int 10(2t+5)^5\,dt \Rightarrow I(2t+5)^5=\frac{5}{6}(2t+5)^6+C$$

5. Divide by the integrating factor:

$$\Rightarrow I=\frac{5}{6}(2t+5)+C(2t+5)^{-5}$$

To find the particular solution, consider the initial conditions, $I(0)=0$:

$$\Rightarrow 0=\frac{5}{6}(2(0)+5)+C(2(0)+5)^{-5}\Rightarrow 0=\frac{25}{6}+\frac{C}{5^5}\Rightarrow \frac{C}{5^5}=-\frac{25}{6}\Rightarrow C=-\frac{25}{6}5^5$$
$$\Rightarrow C=-\frac{78125}{6}$$

Hence, our (particular) solution is $I=\frac{5}{6}(2t+5)-\frac{78125}{6}(2t+5)^{-5}$

8. Solve: $y' = \frac{x-y}{x+y}$

Since the expression for y' seems fairly "symmetric" in terms of x and y, let's try substitution.

Let $v = \frac{y}{x}$. Then $y = vx \Rightarrow y' = \frac{dv}{dx}x + v$

Given the equation, $y' = \frac{x-y}{x+y}$, we can divide top and bottom by x, yielding: $y' = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$.

Now we can substitute our expressions involving v and $\frac{dv}{dx}$ into this equation:

$$\Rightarrow \underbrace{\frac{dv}{dx}x + v}_{y'} = \frac{1-v}{1+v}$$

Separate:

$$\Rightarrow \frac{dv}{dx}x = \frac{1-v}{1+v} - v \Rightarrow \frac{dv}{dx}x = \frac{1-v}{1+v} - \frac{v+v^2}{1+v} \Rightarrow \frac{dv}{dx}x = \frac{1-2v-v^2}{1+v} \Rightarrow \frac{1+v}{1-2v-v^2}dv = \frac{1}{x}dx$$

Integrate:

$$u = 1 - 2v - v^2$$
$$du = (-2 - 2v)\,dv$$
$$-\tfrac{1}{2}du = (1+v)\,dv$$

We have: $\frac{1+v}{1-2v-v^2}dv = \frac{1}{x}dx \Rightarrow \underbrace{\frac{1}{1-2v-v^2}}_{\frac{1}{u}}\underbrace{(1+v)\,dv}_{-\frac{1}{2}du} = \frac{1}{x}dx \Rightarrow \int \frac{1}{u}\left(-\frac{1}{2}du\right) = \int \frac{1}{x}dx$

$$\Rightarrow -\tfrac{1}{2}\int \tfrac{1}{u}du = \int \tfrac{1}{x}dx \Rightarrow -\tfrac{1}{2}\ln|u| = \ln|x| + C \Rightarrow \ln|u| = -2\ln|x| + C$$
$$\Rightarrow \ln|u| = \ln|x|^{-2} + C \Rightarrow e^{\ln|u|} = e^{\ln|x|^{-2}+C} \Rightarrow |u| = e^{\ln|x|^{-2}}e^C \Rightarrow |u| = C_1 e^{\ln|x|^{-2}}$$
$$\Rightarrow |u| = C_1|x|^{-2} \Rightarrow |u| = \frac{C_1}{x^2}$$

Re-express this in terms of v :

$$\Rightarrow |1 - 2v - v^2| = \frac{C_1}{x^2}$$

Now re-express in terms of y :

$$\Rightarrow \left|1 - 2\frac{y}{x} - \left(\frac{y}{x}\right)^2\right| = \frac{C_1}{x^2} \Rightarrow x^2\left|1 - 2\frac{y}{x} - \left(\frac{y}{x}\right)^2\right| = C_1 \Rightarrow |x^2 - 2xy - y^2| = C_1$$

Since C_1 is an arbitrary constant and can be either positive or negative, we can discard the absolute value bars.

$$\Rightarrow x^2 - 2xy - y^2 = C_1$$

Method #2

Observe: $y' = \frac{x-y}{x+y}$

$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y}$

$\Rightarrow (x+y)\,dy = (x-y)\,dx$

$\Rightarrow (x-y)\,dx - (x+y)\,dy = 0$

$\Rightarrow \underbrace{(x-y)}_{M}dx + \underbrace{[-(x+y)]}_{N}dy = 0$

Obeserve: $\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}$

Thus, the equation is exact.

For our solution, we are looking for a function, U such that $\frac{\partial U}{\partial x} = M$ and $\frac{\partial U}{\partial y} = N$.

To find U, integrate:

$U = \int \frac{\partial U}{\partial x}\partial x = \int M \partial x = \int (x-y)\,\partial x = \frac{x^2}{2} - yx + F(y) + C$

Also:

$U = \int \frac{\partial U}{\partial y}\partial y = \int N \partial y = \int -(x+y)\,\partial y = -xy - \frac{y^2}{2} + G(x) + C$

To define U completely (i.e., determine the identity of $F(y)$ and $G(x)$), we must compare the two expressions for U.

$U = \frac{x^2}{2} - yx + F(y) + C = -xy - \frac{y^2}{2} + G(x) + C$

By comparing all sides of the equation, we have $G(x) = \frac{x^2}{2}$ and $F(y) = -\frac{y^2}{2}$

$\Rightarrow U = \frac{x^2}{2} - yx - \frac{y^2}{2} = C_1$