Differential Equations Practice Test #1A - Solutions

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Solutions

1. Solve: $\frac{dy}{dx} = \frac{x}{y}$; $y(0) = 16$

   Separate!

   $\frac{dy}{dx} = \frac{x}{y} \Rightarrow ydy = xdx$

   Integrate!

   $\int ydy = \int xdx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C \Rightarrow y^2 = x^2 + C_1$

   Given the initial condition, $y(0) = 16$, we can solve for C.

   $\Rightarrow 16^2 = 0^2 + C_1 \Rightarrow C_1 = 16^2$

   So our particular solution is $y^2 = x^2 + 16^2$

2. Solve: $xy' = 3x + 2y; \quad y(1) = 1$

We can't separate variables on this one. BUT, there are two different ways that we can solve the equation.

Method #1 -(First Order Linear)

Rewrite as: $y' = 3 + 2x^{-1}y$

$$\Rightarrow y' - 2x^{-1}y = 3$$

$$\Rightarrow y' + \underbrace{\left(-2x^{-1}\right)}_{P(x)}y = \underbrace{3}_{Q(x)}$$

(a) 1. Our integrating factor is $e^{\int P(x)dx} = e^{\int -2x^{-1}dx} = e^{-2\ln(x)} = e^{\ln\left(x^{-2}\right)} = x^{-2}$

2. Next, we multiply both sides of the equation by the integrating factor.

$y'x^{-2} - 2x^{-3}y = 3x^{-2}$

3. The left hand side must be the derivative of a product (this is the entire reason that we multiply both sides by the integrating factor). **That product is always y times the integrating factor.** Rewrite the left hand side as such.

$\Rightarrow \frac{d}{dx}\left[yx^{-2}\right] = 3x^{-2}$

4. Integrate!

$\Rightarrow \int \frac{d}{dx}\left[yx^{-2}\right]dx = \int 3x^{-2}dx$

$\Rightarrow yx^{-2} = -3x^{-1} + C$

5. Divide by the integrating factor

$\Rightarrow y = -3x + Cx^2$

Recall: $y(1) = 1$ We can use this initial condition to find C.

$\Rightarrow 1 = -3(1) + C(1)^2 \Rightarrow C = 4$

So our (particular) solution is $y = 4x^2 - 3x$

Method #2 - (Use the substitution, $v = \frac{y}{x}$)

Given, $xy' = 3x + 2y$, we can divide both sides by x, yielding: $y' = 3 + 2\frac{y}{x}$

If we let $v = \frac{y}{x}$, then $y = vx, \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v$

Substituting into the equation $y' = 3 + 2\frac{y}{x}'$ we have:

$$\frac{dv}{dx}x + v = 3 + 2v$$

Now Separate!

$\Rightarrow \frac{dv}{dx}x = 3 + v \Rightarrow \frac{1}{3+v}x\frac{dv}{dx} = 1 \Rightarrow \frac{1}{3+v}dv = \frac{1}{x}dx$

Integrate!

$\int \frac{1}{3+v}dv = \int \frac{1}{x}dx \Rightarrow \ln(3+v) = \ln(x) + C \Rightarrow e^{\ln(3+v)} = e^{\ln(x)+C} \Rightarrow 3 + v = e^{\ln(x)} \underbrace{e^C}_{\text{constant}} \Rightarrow 3 + v = C_1 e^{\ln(x)} \Rightarrow 3 + v = C_1 x \Rightarrow 3 + \frac{y}{x} = C_1 x \Rightarrow 3x + y = C_1 x^2 \Rightarrow y = C_1 x^2 - 3x$

(a) 1. **Recall:** $y(1) = 1$ We can use this initial condition to find C.

$\Rightarrow 1 = C_1 (1)^2 - 3(1) \Rightarrow C_1 = 4$

So our (particular) solution is $y = 4x^2 - 3x$

3. Solve: $\frac{dy}{dx} - x^2 + 3x = 0$

   Separate!

   $dy = (x^2 - 3x)\, dx$

   Integrate!

   $\int dy = \int (x^2 - 3x)\, dx \Rightarrow y = \frac{x^3}{3} - \frac{3}{2}x^2 + C$

4. Solve: $\frac{dy}{dx} = -\frac{x+y}{x}$; Show that this is an exact differential equation, and solve accordingly.

   Rewrite as

   $$xdy = -(x+y)\, dx \Rightarrow \underbrace{(x+y)}_{M}dx + \underbrace{x}_{N}dy = 0$$

   Observe: $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$ Hence, the equation is exact.

   For our solution, we are looking for a function, U such that $\frac{\partial U}{\partial x} = M$ and $\frac{\partial U}{\partial y} = N$.

   To find U, integrate:

   $U = \int \frac{\partial U}{\partial x}\partial x = \int M\partial x = \int (x+y)\, \partial x = \frac{x^2}{2} + yx + F(y) + C$

   Also:

   $U = \int \frac{\partial U}{\partial y}\partial y = \int N\partial y = \int x\partial y = yx + G(x) + C$

   To define U completely (i.e., determine the identity of $F(y)$ and $G(x)$), we must compare the two expressions for U.

   $U = \frac{x^2}{2} + yx + F(y) + C = yx + G(x) + C$

   By comparing all sides of the equation, we have $G(x) = \frac{x^2}{2}$ and $F(y) = 0$

   $\Rightarrow U = \frac{x^2}{2} + yx + C$

5. Show that $y = A_1 \cos(3x) + B_1 \sin(3x)$ is a solution of the differential equation $y'' + 9y = 0$.

Observe:

$y' = -3A_1 \sin(3x) + 3B_1 \cos(3x)$

$y'' = -9A_1 \cos(3x) - 9B_1 \sin(3x)$

Plugging y, y', and y'' into the differential equation $y'' + 9y = 0$, we have:

$$\underbrace{(-9A_1 \cos(3x) - 9B_1 \sin(3x))}_{y''} + \underbrace{9(A_1 \cos(3x) + B_1 \sin(3x))}_{9y} = 0$$

Thus, $y = A_1 \cos(3x) + B_1 \sin(3x)$ is a solution of the differential equation $y'' + 9y = 0$.

6. Classify the following according to **order** and **linearity.**

(a) $y'' + y' = x$

Order 2 (because y'' is the highest order derivative) and **linear** (because all derivatives are raised to the first power)

(b) $y'' + y = x^2$

Order 2 (because y'' is the highest order derivative) and **linear** (because all derivatives are raised to the first power)

(c) $\frac{dy}{dx} = \frac{x}{1+y^2}$

Order 1 (because $\frac{dy}{dx}$ is the highest order derivative) and **non-linear** (because y is squared and also because y appears in the denominator)

(d) $\frac{dy}{dx} = \frac{y^2}{x^2}$

Order 1 (because $\frac{dy}{dx}$ is the highest order derivative) and **non-linear** (because y is squared)

(e) $y^{(4)} - y''' + xyy' = 0$

Order 4 (because $y^{(4)}$ is the highest order derivative) and **non-linear** (because y and y' appear together in the same term, essentially making the term xyy' have degree 2)

(f) $y^{(4)} - y''' + x^2y' = x^4 - 3$

Order 4 (because $y^{(4)}$ is the highest order derivative) and **linear** (because all derivatives are raised to the first power)

7. Solve: $\frac{dI}{dt} + 5I = 10; \quad I(0) = 0; \quad$ assume that $I < 2$

This can be done by "Separation of Variables."

$\frac{dI}{dt} + 5I = 10 \Rightarrow \frac{dI}{dt} = -5I + 10 \Rightarrow dI = (-5I + 10)\,dt \Rightarrow \frac{dI}{(-5I+10)} = dt$

Now integrate!

$$\int \frac{dI}{(-5I+10)} = \int dt \Rightarrow \int \underbrace{\frac{1}{(-5I+10)}}_{u^{-1}} \underbrace{dI}_{-\frac{1}{5}du} = \int dt \Rightarrow \int u^{-1}\left(-\frac{1}{5}du\right) = \int dt$$

$$\boxed{\begin{array}{rcl} u & = & -5I + 10 \\ \frac{du}{dI} & = & -5 \\ du & = & -5dt \\ -\frac{1}{5}du & = & dt \end{array}}$$

$\Rightarrow -\frac{1}{5}\int u^{-1}du = \int dt \Rightarrow -\frac{1}{5}\ln|u| = t + C \Rightarrow \ln|u| = -5t + C_1 \Rightarrow \ln|-5I + 10| = -5t + C_1 \Rightarrow \ln(-5I + 10) = -5t + C_1 \Rightarrow e^{\ln(-5I+10)} = e^{-5t+C_1}$

$\Rightarrow -5I + 10 = e^{-5t}e^{C_1} \Rightarrow -5I + 10 = C_2e^{-5t} \Rightarrow -5I = C_2e^{-5t} - 10 \Rightarrow I = C_3e^{-5t} + 2$

Since $I(0) = 0$, we have $0 = C_3e^{-5(0)} + 2 \Rightarrow 0 = C_3 + 2 \Rightarrow C_3 = -2$.

Thus, $I = -2e^{-5t} + 2$ is the unique particular solution.

8. Solve: $y' + \frac{y}{x} = 1$ (assume that $x > 0$)

Method #1

This is a Linear, First Order Diff. Eq., $y' + \underbrace{\frac{1}{x}}_{P(x)} y = \underbrace{1}_{Q(x)}$ so we'll solve accordingly.

(a) 1. Our integrating factor is $e^{\int P(x)dx} = e^{\int \frac{1}{2}dx} = e^{\ln(x)} = e^{\ln(x)} = x$

2. Next, we multiply both sides of the equation by the integrating factor.

   $y'x + x\left(\frac{1}{x}y\right) = x \cdot 1 \Rightarrow y'x + y = x$

3. The left hand side must be the derivative of a product (this is the entire reason that we multiply both sides by the integrating factor). **That product is always y times the integrating factor.** Rewrite the left hand side as such.

   $\Rightarrow \frac{d}{dx}[yx] = x$

4. Integrate!

   $\Rightarrow \int \frac{d}{dx}[yx]\,dx = \int x dx$

   $\Rightarrow yx = \frac{1}{2}x^2 + C$

5. Divide by the integrating factor

   $\Rightarrow y = \frac{1}{2}x + cx^{-1}$