

MTH 3311 - Practice Test #1 - Solutions

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1. Test the First-Order, Linear Differential Equation for exactness. If the equation is exact, find the solution.

$$(12x^3 + 4y^3) dx + (6 \cos y + 12xy^2) dy = 0$$

$$\underbrace{(12x^3 + 4y^3)}_{M(x,y)} dx + \underbrace{(6 \cos y + 12xy^2)}_{N(x,y)} dy = 0$$

By convention, we let $M(x, y)$ be the co-factor of dx and we let $N(x, y)$ be the co-factor of dy .

i.e., $M(x, y) = 12x^3 + 4y^3$ and $N(x, y) = 6 \cos y + 12xy^2$

If the Differential equation is **exact**, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 12y^2 = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x, y)$ such that the equation $U(x, y) = C$ relates the solution y implicitly as a function of x .

To find $U(x, y)$, we compute the integrals $\int M(x, y) dx$ and $\int N(x, y) dy$.

$$U(x, y) = \int M(x, y) dx = \int (12x^3 + 4y^3) dx = 3x^4 + 4xy^3 + f(y) + C$$

$$U(x, y) = \int N(x, y) dy = \int (6 \cos(y) + 12xy^2) dy = 6 \sin(y) + 4xy^3 + g(x) + C$$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x, y) dx$ and $\int N(x, y) dy$.

$$\begin{array}{ccccccc} U(x, y) & = & 3x^4 & + & 4xy^3 & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ U(x, y) & = & g(x) & + & 4xy^3 & + & 6 \sin(y) & + & C \end{array}$$

Thus, $f(y) = 6 \sin(y)$ and $g(x) = 3x^4$

Our solution $y = y(x)$ is given implicitly by the equation $U(x, y) = C$

$$3x^4 + 4xy^3 + 6 \sin(y) = C$$

2. Solve the differential equation $\frac{dy}{dx} = x^2 + \frac{x^2}{y}$, subject to the initial condition $y(3) = 0$ (Assume that $y \geq 0$)

Let's see if we can separate the variables.

$$\frac{dy}{dx} = x^2 + \frac{x^2}{y} \Rightarrow \frac{dy}{dx} = x^2 \left(1 + \frac{1}{y}\right) \Rightarrow \frac{dy}{dx} = x^2 \left(\frac{y+1}{y}\right) \Rightarrow \frac{y}{y+1} dy = x^2 dx$$

Integrate:

$$\int \frac{y}{y+1} dy = \int x^2 dx$$

Scratchwork:

$$\int \frac{y}{y+1} dy = \int \left(\frac{y+1}{y+1} - \frac{1}{y+1}\right) dy = \int \left(1 - \frac{1}{y+1}\right) dy = y - \ln(y+1) + C$$

Back to our equation:

$$\int \frac{y}{y+1} dy = \int x^2 dx \Rightarrow y - \ln(y+1) + C = \frac{1}{3}x^3 \Rightarrow y - \ln(y+1) - \frac{1}{3}x^3 = C$$

Incorporating the initial condition, $y(3) = 0$, we have:

$$(0) - \ln((0) + 1) - \frac{1}{3}(3)^3 = C$$

$$\Rightarrow C = -9$$

Our solution y is expressed implicitly by the equation:

$$y - \ln(y+1) - \frac{1}{3}x^3 = -9$$

3. Solve the differential equation $y' - y = -x^{-3}e^x$.

Rewriting the equation slightly as:

$$y' + (-1)y = -x^{-3}e^x,$$

This fits the form:

$$y' + P(x)y = Q(x), \quad \text{with } P(x) = -1, \text{ and } Q(x) = -x^{-3}e^x$$

We can solve this using the “Integrating Factor Method”

1. Compute the integrating factor:

$$e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$$

2. Multiply both sides by the integrating factor

$$y'e^{-x} - e^{-x}y = (-x^{-3}e^x)e^{-x}$$

3. Express the left hand side as the derivative of a product

$$\Rightarrow \frac{d}{dx} [e^{-x}y] = -x^{-3}$$

4. Integrate both sides w.r.t. x .

$$\int \frac{d}{dx} [e^{-x}y] dx = \int (-x^{-3}) dx \Rightarrow e^{-x}y = \int (-x^{-3}) dx = \frac{1}{2}x^{-2} + C$$

$$\text{i.e., } e^{-x}y = \frac{1}{2}x^{-2} + C$$

$$\Rightarrow y = e^x \left(\frac{1}{2}x^{-2} + C \right)$$

Our solution y is given by the equation:

$$\boxed{y = e^x \left(\frac{1}{2}x^{-2} + C \right)}$$

4. Solve the differential equation $(xy + 4y^2 + 9x^2) dx - x^2 dy = 0$

We might be able to re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$$(xy + 4y^2 + 9x^2) dx - x^2 dy = 0$$

$$\Rightarrow (xy + 4y^2 + 9x^2) - x^2 \frac{dy}{dx} = 0$$

$$\Rightarrow -x^2 \frac{dy}{dx} = -xy - 4y^2 - 9x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{4y^2}{x^2} + 9$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + 4\left(\frac{y}{x}\right)^2 + 9 \quad (\text{i.e., } \frac{dy}{dx} = f\left(\frac{y}{x}\right))$$

$$\text{let } v = \frac{y}{x} \quad (\text{i.e., } y = vx) \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting, we have:

$$v + x \frac{dv}{dx} = v + 4v^2 + 9 \quad \text{Now Separate!}$$

$$\Rightarrow x \frac{dv}{dx} = 4v^2 + 9$$

$$\Rightarrow \frac{1}{4v^2+9} dv = \frac{1}{x} dx \quad \text{Now Integrate!}$$

$$\int \frac{1}{4v^2+9} dv = \int \frac{1}{x} dx$$

Scratchwork: Let $a = 3$; $u = 2v$; $du = 2dv$; $\frac{1}{2} du = dv$

$$\int \frac{1}{4v^2+9} dv = \int \frac{1}{u^2+a^2} \frac{1}{2} du = \frac{1}{2} \frac{1}{a} \arctan\left(\frac{u}{a}\right) = \frac{1}{6} \arctan\left(\frac{2v}{3}\right)$$

Back to our Integrals:

$$\int \frac{1}{4v^2+9} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{1}{6} \arctan\left(\frac{2v}{3}\right) = \ln|x| + C \Rightarrow \arctan\left(\frac{2v}{3}\right) = \ln(x^6) + C \Rightarrow \arctan\left(\frac{2\left(\frac{y}{x}\right)}{3}\right) = \ln(x^6) + C$$

$$\Rightarrow \arctan\left(\frac{2y}{3x}\right) = \ln(x^6) + C$$

Our solution y is given implicitly by the equation:

$$\arctan\left(\frac{2y}{3x}\right) = \ln(x^6) + C$$

5. Test the First-Order, Linear Differential Equation for exactness. If the equation is exact, find the solution.

$$(16x^3y^2 + 12xy^2) dx + (8yx^4 + 12yx^2 - 3e^{-y}) dy = 0$$

$$\underbrace{(16x^3y^2 + 12xy^2)}_{M(x,y)} dx + \underbrace{(8yx^4 + 12yx^2 - 3e^{-y})}_{N(x,y)} dy = 0$$

By convention, we let $M(x, y)$ be the co-factor of dx and we let $N(x, y)$ be the co-factor of dy .

i.e., $M(x, y) = 16x^3y^2 + 12xy^2$ and $N(x, y) = 8yx^4 + 12yx^2 - 3e^{-y}$

If the Differential equation is **exact**, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 32yx^3 + 24yx = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x, y)$ such that the equation $U(x, y) = C$ relates the solution y implicitly as a function of x .

To find $U(x, y)$, we compute the integrals $\int M(x, y) dx$ and $\int N(x, y) dy$.

$$U(x, y) = \int M(x, y) dx = \int (16x^3y^2 + 12xy^2) dx = 4x^4y^2 + 6x^2y^2 + f(y) + C$$

$$U(x, y) = \int N(x, y) dy = \int (8yx^4 + 12yx^2 - 3e^{-y}) dy = 3e^{-y} + 6x^2y^2 + 4x^4y^2 + g(x) + C$$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x, y) dx$ and $\int N(x, y) dy$.

$$\begin{array}{rcccccccc} U(x, y) & = & 0 & + & 4x^4y^2 & + & 6x^2y^2 & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ U(x, y) & = & g(x) & + & 4x^4y^2 & + & 6x^2y^2 & + & 3e^{-y} & + & C \end{array}$$

Thus, $f(y) = 3e^{-y}$ and $g(x) = 0$

Our solution $y = y(x)$ is given implicitly by the equation $U(x, y) = C$

$$4x^4y^2 + 6x^2y^2 + 3e^{-y} = C$$

6. Solve the differential equation $\frac{dy}{dx} = \frac{x+2}{y^2}$, subject to the initial condition $y(2) = 3$

Let's see if we can separate the variables.

$$\frac{dy}{dx} = \frac{x+2}{y^2} \Rightarrow y^2 dy = (x+2) dx \quad (\text{Yipes! I didn't expect it to be THAT easy!})$$

Integrate:

$$\int y^2 dy = \int (x+2) dx \Rightarrow \frac{1}{3}y^3 = \frac{1}{2}x^2 + 2x + C \Rightarrow y^3 = \frac{3}{2}x^2 + 6x + C$$

Incorporating the initial condition, $y(2) = 3$, we have:

$$(3)^3 = \frac{3}{2}(2)^2 + 6(2) + C \Rightarrow 27 = 18 + C$$

$$\Rightarrow C = 9$$

$$\text{Thus, } y^3 = \frac{3}{2}x^2 + 6x + 9 \Rightarrow y = \left(\frac{3}{2}x^2 + 6x + 9\right)^{\frac{1}{3}}$$

Our solution y is given by:

$$y = \left(\frac{3}{2}x^2 + 6x + 9\right)^{\frac{1}{3}}$$

7. Solve the differential equation $y' + \cot(x)y = x$ (Assume that $0 < x < \frac{\pi}{2}$)

This fits the form:

$$y' + P(x)y = Q(x), \quad \text{with } P(x) = \cot(x), \text{ and } Q(x) = x$$

We can solve this using the “Integrating Factor Method”

1. Compute the integrating factor:

$$e^{\int P(x)dx} = e^{\int \cot(x)dx} = e^{\ln(\sin(x))} = \sin(x)$$

2. Multiply both sides by the integrating factor

$$y' \sin(x) + \cot(x) \sin(x) y = x \sin(x)$$

3. Express the left hand side as the derivative of a product

$$\Rightarrow \frac{d}{dx} [\sin(x) y] = x \sin(x)$$

4. Integrate both sides w.r.t. x .

$$\int \left(\frac{d}{dx} [\sin(x) y] \right) dx = \int x \sin(x) dx$$

$$\Rightarrow \sin(x) y = \int \underbrace{x}_u \underbrace{\sin(x)dx}_{dv} = \underbrace{x}_u \underbrace{(-\cos(x))}_v - \int \underbrace{(-\cos(x))}_v \underbrace{dx}_{du} = -x \cos(x) + \sin(x) + C$$

$$\Rightarrow \sin(x) y = -x \cos(x) + \sin(x) + C \Rightarrow y = -x \cot(x) + 1 + C \csc(x)$$

i.e., $y = -x \cot(x) + 1 + C \csc(x)$

Our solution y is given by the equation:

$$y = -x \cot(x) + 1 + C \csc(x)$$

8. Solve the differential equation $(xe^{\frac{y}{x}} - y) dx = -x dy$

We might be able to re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$$(xe^{\frac{y}{x}} - y) dx = -x dy$$

$$\Rightarrow (xe^{\frac{y}{x}} - y) = -x \frac{dy}{dx}$$

$$\Rightarrow (e^{\frac{y}{x}} - \frac{y}{x}) = -\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x} - e^{\frac{y}{x}}\right) \quad (\text{i.e., } \frac{dy}{dx} = f\left(\frac{y}{x}\right))$$

$$\text{let } v = \frac{y}{x} \quad (\text{i.e., } y = vx) \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting into the equation $\frac{dy}{dx} = \left(\frac{y}{x} - e^{\frac{y}{x}}\right)$, we have:

$$v + x \frac{dv}{dx} = v - e^v \quad \text{Now Separate!}$$

$$\Rightarrow x \frac{dv}{dx} = -e^v$$

$$\Rightarrow -x \frac{dv}{dx} = e^v$$

$$e^{-v} dv = -\frac{1}{x} dx \quad \text{Now Integrate!}$$

$$\int e^{-v} dv = -\int \frac{1}{x} dx \Rightarrow -e^{-v} = -\ln|x| + C \Rightarrow e^{-v} = \ln|x| + C_1$$

Now re-express in terms of x and y

$$\Rightarrow e^{-\frac{y}{x}} = \ln|x| + C_1$$

Our solution y is given implicitly by the equation:

$$e^{-\frac{y}{x}} = \ln|x| + C$$

9. Test the First-Order, Linear Differential Equation for exactness. If the equation is exact, find the solution.

$$(4x^3y^2 + 2xy^2) dx + (8x^4 + 12x^2) dy = 0$$

$$\underbrace{(4x^3y^2 + 2xy^2)}_{M(x,y)} dx + \underbrace{(8x^4 + 12x^2)}_{N(x,y)} dy = 0$$

By convention, we let $M(x, y)$ be the co-factor of dx and we let $N(x, y)$ be the co-factor of dy .

i.e., $M(x, y) = 4x^3y^2 + 2xy^2$ and $N(x, y) = 8x^4 + 12x^2$

If the Differential equation is **exact**, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 8yx^3 + 4yx \neq 32x^3 + 24x = \frac{\partial N}{\partial x}$

| |
|---------------------------|
| The equation is NOT Exact |
|---------------------------|

10. Solve the differential equation $\frac{dy}{dx} = \frac{2y+1}{x-3}$, subject to the initial condition $y(4) = 1$ (Assume that $x, y > 0$)

Let's see if we can separate the variables.

$$\frac{dy}{dx} = \frac{2y+1}{x-3} \Rightarrow \frac{1}{2y+1} dy = \frac{1}{x-3} dx \quad (\text{Yipes - Another easy one!})$$

Integrate:

$$\int \frac{1}{2y+1} dy = \int \frac{1}{x-3} dx$$

Scratchwork:

$$\int \underbrace{\frac{1}{2y+1}}_{\frac{1}{u}} \underbrace{dy}_{\frac{1}{2} du} = \int \frac{1}{u} \frac{1}{2} du = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln(u) + C = \frac{1}{2} \ln(2y+1) + C$$

Back to our equation:

$$\int \frac{1}{2y+1} dy = \int \frac{1}{x-3} dx \Rightarrow \frac{1}{2} \ln(2y+1) = \ln(x-3) + C \Rightarrow \ln(2y+1) = 2 \ln(x-3) + C$$

$$\Rightarrow \ln(2y+1) = \ln(x-3)^2 + C \Rightarrow e^{\ln(2y+1)} = e^{\ln(x-3)^2 + C} \Rightarrow 2y+1 = e^{\ln(x-3)^2} e^C$$

$$\Rightarrow y + \frac{1}{2} = C_1 e^{\ln(x-3)^2} \Rightarrow y = C_1 e^{\ln(x-3)^2} - \frac{1}{2} \Rightarrow y = C_1 (x-3)^2 - \frac{1}{2}$$

$$\text{i.e., } y = C_1 (x-3)^2 - \frac{1}{2}$$

Incorporating the initial condition, $y(4) = 1$, we have:

$$(1) = C_1 (4-3)^2 - \frac{1}{2}$$

$$\Rightarrow C_1 = \frac{3}{2}$$

Our solution y is expressed implicitly by the equation:

$$y = \frac{3}{2} (x-3)^2 - \frac{1}{2}$$

11. Solve the differential equation $xy' + y = x^4 + x^2$ (Assume that $x > 0$)

Rewriting the equation as:

$$y' + \frac{1}{x}y = x^3 + x,$$

This fits the form:

$$y' + P(x)y = Q(x), \quad \text{with } P(x) = \frac{1}{x}, \text{ and } Q(x) = x^3 + x$$

We can solve this using the “Integrating Factor Method”

1. Compute the integrating factor:

$$e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$$

2. Multiply both sides by the integrating factor

$$y'x + x\frac{1}{x}y = x(x^3 + x)$$

$$\text{i.e., } y'x + 1y = x^4 + x^2$$

3. Express the left hand side as the derivative of a product

$$\frac{d}{dx}[xy] = x^4 + x^2$$

4. Integrate both sides w.r.t. x

$$\Rightarrow \int \left(\frac{d}{dx}[xy]\right) dx = \int (x^4 + x^2) dx$$

$$\Rightarrow xy = \frac{1}{5}x^5 + \frac{1}{3}x^3 + C \Rightarrow y = \frac{1}{5}x^4 + \frac{1}{3}x^2 + Cx^{-1}$$

$$\text{i.e., } y = \frac{1}{5}x^4 + \frac{1}{3}x^2 + Cx^{-1}$$

Our solution y is given by the equation:

$$y = \frac{1}{5}x^4 + \frac{1}{3}x^2 + Cx^{-1}$$

12. Solve the differential equation $xydx - (x^2 - y^2) dy = 0$ (Assume that $x, y > 0$)

We might be able to re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$$xydx - (x^2 - y^2) dy = 0$$

$$\Rightarrow xy - (x^2 - y^2) \frac{dy}{dx} = 0$$

$$\Rightarrow -(x^2 - y^2) \frac{dy}{dx} = -xy$$

$$\frac{dy}{dx} = \frac{xy}{(x^2 - y^2)}$$

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{1 - \left(\frac{y}{x}\right)^2} \quad (\text{i.e., } \frac{dy}{dx} = f\left(\frac{y}{x}\right))$$

$$\text{let } v = \frac{y}{x} \quad (\text{i.e., } y = vx) \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting into the equation $\frac{dy}{dx} = \frac{\frac{y}{x}}{1 - \left(\frac{y}{x}\right)^2}$, we have:

$$v + x \frac{dv}{dx} = \frac{v}{1 - v^2} \quad \text{Now Separate!}$$

$$x \frac{dv}{dx} = \frac{v}{1 - v^2} - v = \frac{v}{1 - v^2} - \frac{v - v^3}{1 - v^2} = \frac{v^3}{1 - v^2}$$

$$\text{i.e., } x \frac{dv}{dx} = \frac{v^3}{1 - v^2}$$

$$\Rightarrow \frac{1 - v^2}{v^3} dv = \frac{1}{x} dx \quad \text{Now Integrate!}$$

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{1}{x} dx$$

Scratchwork:

$$\int \frac{1 - v^2}{v^3} dv = \int \left(\frac{1}{v^3} - \frac{v^2}{v^3} \right) dv = \int \left(v^{-3} - \frac{1}{v} \right) dv = \frac{v^{-2}}{-2} - \ln(v) = -\frac{1}{2v^2} - \ln(v) + C$$

Back to our equation:

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{1}{x} dx$$

$$\Rightarrow -\frac{1}{2v^2} - \ln(v) + C = \ln(x)$$

Re-express in terms of x and y .

$$\Rightarrow -\frac{1}{2\left(\frac{y}{x}\right)^2} - \ln\left(\frac{y}{x}\right) + C = \ln(x)$$

$$\Rightarrow -\frac{1}{2\left(\frac{y}{x}\right)^2} - (\ln(y) - \ln(x)) + C = \ln(x)$$

$$\Rightarrow -\frac{1}{2\left(\frac{y}{x}\right)^2} - \ln(y) + \ln(x) + C = \ln(x)$$

$$\Rightarrow -\frac{1}{2\left(\frac{y}{x}\right)^2} - \ln(y) + C = 0$$

$$\Rightarrow \frac{x^2}{2y^2} + \ln(y) = C$$

Our solution y is given implicitly by the equation:

$$\frac{x^2}{2y^2} + \ln(y) = C$$

13. Test the First-Order, Linear Differential Equation for exactness. If the equation is exact, find the solution.

$$(\cos(x) + 3ye^{xy}) dx + (\tan^2(y) + 3xe^{xy} + 1) dy = 0$$

$$\underbrace{(\cos(x) + 3ye^{xy})}_{M(x,y)} dx + \underbrace{(\tan^2(y) + 3xe^{xy} + 1)}_{N(x,y)} dy = 0$$

By convention, we let $M(x, y)$ be the co-factor of dx and we let $N(x, y)$ be the co-factor of dy .

i.e., $M(x, y) = \cos(x) + 3ye^{xy}$ and $N(x, y) = \tan^2(y) + 3xe^{xy} + 1$

If the Differential equation is **exact**, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 3e^{xy} + 3xye^{xy} = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x, y)$ such that the equation $U(x, y) = C$ relates the solution y implicitly as a function of x .

To find $U(x, y)$, we compute the integrals $\int M(x, y) dx$ and $\int N(x, y) dy$.

$$U(x, y) = \int M(x, y) dx = \int (\cos(x) + 3ye^{xy}) dx = \sin(x) + 3e^{xy} + f(y) + C$$

$$U(x, y) = \int N(x, y) dy = \int (\tan^2(y) + 3xe^{xy} + 1) dy = \tan(y) + 3e^{xy} + g(x) + C$$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x, y) dx$ and $\int N(x, y) dy$.

$$\begin{array}{ccccccc} U(x, y) & = & \sin(x) & + & 3e^{xy} & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ U(x, y) & = & g(x) & + & 3e^{xy} & + & \tan(y) & + & C \end{array}$$

Thus, $f(y) = \tan(y)$ and $g(x) = \sin(x)$

Our solution $y = y(x)$ is given implicitly by the equation $U(x, y) = C$

$$\sin(x) + 3e^{xy} + \tan(y) = C$$

14. Solve the differential equation $\frac{dy}{dx} = x - xy - y + 1$. (Assume that $y > 1$)

Let's see if we can separate the variables.

$$\begin{aligned}\frac{dy}{dx} &= x - xy - y + 1 \Rightarrow \frac{dy}{dx} = x(1-y) + (1-y) \Rightarrow \frac{dy}{dx} = (x+1)(1-y) \\ \Rightarrow \frac{1}{1-y} dy &= (x+1) dx\end{aligned}$$

Integrate:

$$\begin{aligned}\int \frac{1}{1-y} dy &= \int (x+1) dx \Rightarrow -\int \frac{1}{y-1} dy = \int (x+1) dx \Rightarrow -\ln(y-1) = \frac{1}{2}x^2 + x + C \\ \Rightarrow \ln(y-1) &= -\frac{1}{2}x^2 - x + C_1 \\ \Rightarrow e^{\ln(y-1)} &= e^{-\frac{1}{2}x^2 - x + C_1} \Rightarrow (y-1) = e^{-\frac{1}{2}x^2 - x} e^{C_1} \Rightarrow (y-1) = C_2 e^{-\frac{1}{2}x^2 - x} \\ \Rightarrow y &= C_2 e^{-\frac{1}{2}x^2 - x} + 1\end{aligned}$$

$$\boxed{y = C_2 e^{-\frac{1}{2}x^2 - x} + 1}$$

Alternatively:

Let $u = 1 - y$

$$\Rightarrow \frac{du}{dy} = -1$$

$$\Rightarrow du = -dy$$

$$\Rightarrow -du = dy$$

$$\int \underbrace{\frac{1}{1-y}}_{\frac{1}{u}} \underbrace{dy}_{-du} = \int (x+1) dx \Rightarrow \int \frac{1}{u} (-du) = \int (x+1) dx \Rightarrow -\int \frac{1}{u} du = \int (x+1) dx$$

$$\Rightarrow -\ln|u| = \frac{1}{2}x^2 + x + C \Rightarrow -\ln|1-y| = \frac{1}{2}x^2 + x + C \Rightarrow \ln|1-y| = -\frac{1}{2}x^2 - x + C_1 \quad (\text{Eq. 1})$$

We are told at the beginning of the exercise that $y > 1$

Hence, $0 > 1 - y$

$$\Rightarrow |1-y| = -(1-y) = (y-1)$$

$$\text{i.e., } |1-y| = (y-1)$$

$$\text{Recall Eq. 1: } \ln|1-y| = -\frac{1}{2}x^2 - x + C_1$$

$$\text{Thus, Eq. 1 can be rewritten: } \ln(y-1) = -\frac{1}{2}x^2 - x + C_1$$

$$\Rightarrow e^{\ln(y-1)} = e^{-\frac{1}{2}x^2 - x + C_1} \Rightarrow (y-1) = e^{-\frac{1}{2}x^2 - x} e^{C_1} \Rightarrow (y-1) = C_2 e^{-\frac{1}{2}x^2 - x}$$

$$\Rightarrow y = C_2 e^{-\frac{1}{2}x^2 - x} + 1$$

$$\boxed{y = C_2 e^{-\frac{1}{2}x^2 - x} + 1}$$

15. Solve the differential equation $\cos(x)y' + y = 10$ Assume that $0 < x < \frac{\pi}{2}$

Rewriting the equation as:

$$y' + \sec(x)y = 10 \sec(x),$$

This fits the form:

$$y' + P(x)y = Q(x), \quad \text{with } P(x) = \sec(x), \text{ and } Q(x) = 10 \sec(x)$$

We can solve this using the “Integrating Factor Method”

1. Compute the integrating factor:

$$e^{\int P(x)dx} = e^{\int \sec(x)dx} = e^{\ln(\sec(x)+\tan(x))} = \sec(x) + \tan(x)$$

2. Multiply both sides by the integrating factor

$$y'(\sec(x) + \tan(x)) + (\sec(x) + \tan(x))\sec(x)y = 10 \sec(x)(\sec(x) + \tan(x))$$

$$\Rightarrow y'(\sec(x) + \tan(x)) + (\sec^2(x) + \sec(x)\tan(x))y = (10 \sec^2(x) + 10 \sec(x)\tan(x))$$

3. Express the left hand side as the derivative of a product

$$\Rightarrow \frac{d}{dx}[(\sec(x) + \tan(x))y] = (10 \sec^2(x) + 10 \sec(x)\tan(x))$$

4. Integrate both sides w.r.t. x

$$\Rightarrow \int \left(\frac{d}{dx}[(\sec(x) + \tan(x))y]\right) dx = \int (10 \sec^2(x) + 10 \sec(x)\tan(x)) dx$$

$$\Rightarrow (\sec(x) + \tan(x))y = 10 \tan(x) + 10 \sec(x) + C$$

$$\text{i.e., } y = 10 + \frac{C}{(\sec(x)+\tan(x))} \Rightarrow y = 10 + \frac{C}{(\sec(x)+\tan(x))} \frac{(\sec(x)-\tan(x))}{\sec(x)-\tan(x)} \Rightarrow y = 10 + C(\sec(x) - \tan(x))$$

$$\text{i.e., } y = 10 + C(\sec(x) - \tan(x))$$

Our solution y is given by the equation:

$$y = 10 + C(\sec(x) - \tan(x))$$

Alternative Solution appears on the next page.

Alternative Solution:

Alternatively, we can solve this equation using Separation of Variables.

Given: $\cos(x)y' + y = 10$ And assuming that $0 < x < \frac{\pi}{2}$,

We can rewrite the equation as:

$$y' + \sec(x)y = 10 \sec(x)$$

Let's see if we can separate the variables.

$$\Rightarrow y' = 10 \sec(x) - \sec(x)y$$

$$\Rightarrow y' = (10 - y) \sec(x)$$

$$\Rightarrow \frac{dy}{dx} = (10 - y) \sec(x)$$

$$\Rightarrow \frac{1}{10-y} dy = \sec(x) dx \quad \text{Integrate!}$$

$$\int \frac{1}{10-y} dy = \int \sec(x) dx$$

Scratchwork:

$$\int \underbrace{\frac{1}{10-y}}_{\frac{1}{u}} \underbrace{dy}_{-du} = \int \frac{1}{u} (-du) = - \int \frac{1}{u} du = - \ln |u| + C = - \ln |10 - y| + C$$

Back to our integrals:

$$\int \frac{1}{10-y} dy = \int \sec(x) dx$$

$$\Rightarrow - \ln |10 - y| = \ln |\sec(x) + \tan(x)| + C$$

(Since $0 < x < \frac{\pi}{2}$, we can discard absolute value bars on the right side of the equation.)

$$\Rightarrow - \ln |10 - y| = \ln (\sec(x) + \tan(x)) + C$$

$$\Rightarrow e^{-\ln |10-y|} = e^{\ln(\sec(x)+\tan(x))+C} = e^{\ln(\sec(x)+\tan(x))} e^C = C (\sec(x) + \tan(x))$$

$$\Rightarrow e^{\ln |10-y|^{-1}} = C (\sec(x) + \tan(x))$$

$$\Rightarrow \frac{1}{|10-y|} = C (\sec(x) + \tan(x))$$

(Since C can be positive or negative, we don't need the remaining absolute value bars)

$$\Rightarrow (10 - y) = C_1 \frac{1}{(\sec(x)+\tan(x))} = C_1 \frac{1}{(\sec(x)+\tan(x))} \frac{(\sec(x)-\tan(x))}{(\sec(x)-\tan(x))} = C_1 \frac{\sec(x)-\tan(x)}{\sec^2(x)-\tan^2(x)} = C_1 \frac{\sec(x)-\tan(x)}{1}$$

$$\text{i.e., } (10 - y) = C_1 (\sec(x) - \tan(x))$$

$$\Rightarrow y - 10 = C_2 (\sec(x) - \tan(x))$$

Our solution y is given by the equation:

$$y = 10 + C (\sec(x) - \tan(x))$$

16. Solve the differential equation $(x^3 + y^3) dx - xy^2 dy = 0$; $y(1) = 0$ (Assume that $x > 0$)

We might be able to re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$$(x^3 + y^3) dx - xy^2 dy = 0$$

$$\Rightarrow (x^3 + y^3) - xy^2 \frac{dy}{dx} = 0$$

$$\Rightarrow -xy^2 \frac{dy}{dx} = -(x^3 + y^3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^3 + y^3)}{xy^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left(1 + \frac{y^3}{x^3}\right)}{x \frac{y^2}{x^3}} = \frac{\left(1 + \frac{y^3}{x^3}\right)}{\frac{y^2}{x^2}} = \frac{1 + \left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$$

i.e., $\frac{dy}{dx} = \frac{1 + \left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$ Now Separate!

$$\text{let } v = \frac{y}{x} \text{ (i.e., } y = vx) \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{1 + v^3}{v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + v^3}{v^2} - v = \frac{1 + v^3}{v^2} - v = \frac{1 + v^3}{v^2} - \frac{v^3}{v^2} = \frac{1}{v^2} = v^{-2}$$

$$\text{i.e., } x \frac{dv}{dx} = v^{-2}$$

$$\Rightarrow v^2 dv = \frac{1}{x} dx \text{ Now Integrate!}$$

$$\Rightarrow \int v^2 dv = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{1}{3} v^3 = \ln(x) + C$$

Re-express in terms of x and y

$$\Rightarrow \frac{1}{3} \left(\frac{y}{x}\right)^3 = \ln(x) + C$$

$$\Rightarrow \frac{1}{3} \frac{y^3}{x^3} = \ln(x) + C$$

$$\Rightarrow y^3 = 3x^3 \ln(x) + C$$

Our solution y is given by the equation:

$$\boxed{y = (3x^3 \ln(x) + C)^{\frac{1}{3}}}$$