MTH 3311 - Practice Test #1 - Solutions

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1. Test the First-Order, Linear Differential Equation for exactness. If the equation is exact, find the solution.

$(12x^3 + 4y^3)\,dx + (6\cos y + 12xy^2)\,dy = 0$

$\underbrace{(12x^3 + 4y^3)}_{M(x,y)}dx + \underbrace{(6\cos y + 12xy^2)}_{N(x,y)}dy = 0$

By convention, we let $M(x,y)$ be the co-factor of dx and we let $N(x,y)$ be the co-factor of dy.

i.e., $M(x,y) = 12x^3 + 4y^3$ and $N(x,y) = 6\cos y + 12xy^2$

If the Differential equation is **exact,** then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 12y^2 = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x,y)$ such that the equation $U(x,y) = C$ relates the solution y implicitly as a function of x.

To find $U(x,y)$, we compute the integrals $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$U(x,y) = \int M(x,y)\,dx = \int (12x^3 + 4y^3)\,dx = 3x^4 + 4xy^3 + f(y) + C$

$U(x,y) = \int N(x,y)\,dy = \int (6\cos(y) + 12xy^2)\,dy = 6\sin(y) + 4xy^3 + g(x) + C$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$$\begin{array}{ccccccccc} U(x,y) & = & 3x^4 & + & 4xy^3 & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ U(x,y) & = & g(x) & + & 4xy^3 & + & 6\sin(y) & + & C \end{array}$$

Thus, $f(y) = 6\sin(y)$ and $g(x) = 3x^4$

Our solution $y = y(x)$ is given implicitly by the equation $U(x,y) = C$

$$\boxed{3x^4 + 4xy^3 + 6\sin(y) = C}$$

2. Solve the differential equation $\frac{dy}{dx} = x^2 + \frac{x^2}{y}$, subject to the initial condition $y(3) = 0$ (Assume that $y \geq 0$)

Let's see if we can separate the variables.

$\frac{dy}{dx} = x^2 + \frac{x^2}{y} \Rightarrow \frac{dy}{dx} = x^2\left(1 + \frac{1}{y}\right) \Rightarrow \frac{dy}{dx} = x^2\left(\frac{y+1}{y}\right) \Rightarrow \frac{y}{y+1}dy = x^2 dx$

Integrate:

$\int \frac{y}{y+1}dy = \int x^2 dx$

Scratchwork:

$\int \frac{y}{y+1}dy = \int \left(\frac{y+1}{y+1} - \frac{1}{y+1}\right) dy = \int \left(1 - \frac{1}{y+1}\right) dy = y - \ln(y+1) + C$

Back to our equation:

$\int \frac{y}{y+1}dy = \int x^2 dx \Rightarrow y - \ln(y+1) + C = \frac{1}{3}x^3 \Rightarrow y - \ln(y+1) - \frac{1}{3}x^3 = C$

Incorporating the initial condition, $y(3) = 0$, we have:

$(0) - \ln((0)+1) - \frac{1}{3}(3)^3 = C$

$\Rightarrow C = -9$

Our solution y is expressed implicitly by the equation:

$y - \ln(y+1) - \frac{1}{3}x^3 = -9$

3. Solve the differential equation $y' - y = -x^{-3}e^x$.

Rewriting the equation slightly as:

$$y' + (-1)\,y = -x^{-3}e^x,$$

This fits the form:

$$y' + P(x)\,y = Q(x)\,, \quad \text{with } P(x) = -1, \text{ and } Q(x) = -x^{-3}e^x$$

We can solve this using the "Integrating Factor Method"

1. Compute the integrating factor:

$e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$

2. Multiply both sides by the integrating factor

$y'e^{-x} - e^{-x}y = \left(-x^{-3}e^x\right)e^{-x}$

3. Express the left hand side as the derivative of a product

$\Rightarrow \frac{d}{dx}\left[e^{-x}y\right] = -x^{-3}$

4. Integrate both sides w.r.t. x.

$\int \frac{d}{dx}\left[e^{-x}y\right]dx = \int \left(-x^{-3}\right)dx \Rightarrow e^{-x}y = \int \left(-x^{-3}\right)dx = \frac{1}{2}x^{-2} + C$

i.e., $e^{-x}y = \frac{1}{2}x^{-2} + C$

$\Rightarrow y = e^x\left(\frac{1}{2}x^{-2} + C\right)$

Our solution y is given by the equation:

$$\boxed{y = e^x\left(\frac{1}{2}x^{-2} + C\right)}$$

4. Solve the differential equation $\left(xy+4y^2+9x^2\right)dx - x^2dy = 0$

We might be able to re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$\left(xy+4y^2+9x^2\right)dx - x^2dy = 0$

$\Rightarrow \left(xy+4y^2+9x^2\right) - x^2\frac{dy}{dx} = 0$

$\Rightarrow -x^2\frac{dy}{dx} = -xy - 4y^2 - 9x^2$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{4y^2}{x^2} + 9$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + 4\left(\frac{y}{x}\right)^2 + 9$ (i.e., $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$)

let $v = \frac{y}{x}$ (i.e., $y = vx$) $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting, we have:

$v + x\frac{dv}{dx} = v + 4v^2 + 9$ Now Separate!

$\Rightarrow x\frac{dv}{dx} = 4v^2 + 9$

$\Rightarrow \frac{1}{4v^2+9}dv = \frac{1}{x}dx$ Now Integrate!

$\int \frac{1}{4v^2+9}dv = \int \frac{1}{x}dx$

Scratchwork: Let $a = 3$; $u = 2v$; $du = 2dv$; $\frac{1}{2}du = dv$

$\int \frac{1}{4v^2+9}dv = \int \frac{1}{u^2+a^2}\frac{1}{2}du = \frac{1}{2}\frac{1}{a}\arctan\left(\frac{u}{a}\right) = \frac{1}{6}\arctan\left(\frac{2v}{3}\right)$

Back to our Integrals:

$\int \frac{1}{4v^2+9}dv = \int \frac{1}{x}dx$

$\Rightarrow \frac{1}{6}\arctan\left(\frac{2v}{3}\right) = \ln|x| + C \Rightarrow \arctan\left(\frac{2v}{3}\right) = \ln\left(x^6\right) + C \Rightarrow \arctan\left(\frac{2\left(\frac{y}{x}\right)}{3}\right) = \ln\left(x^6\right) + C$

$\Rightarrow \arctan\left(\frac{2y}{3x}\right) = \ln\left(x^6\right) + C$

Our solution y is given implicitly by the equation:

$$\boxed{\arctan\left(\frac{2y}{3x}\right) = \ln\left(x^6\right) + C}$$

5. Test the First-Order, Linear Differential Equation for exactness. If the equation is exact, find the solution.

$$\left(16x^3y^2 + 12xy^2\right) dx + \left(8yx^4 + 12yx^2 - 3e^{-y}\right) dy = 0$$

$$\underbrace{\left(16x^3y^2 + 12xy^2\right)}_{M(x,y)} dx + \underbrace{\left(8yx^4 + 12yx^2 - 3e^{-y}\right)}_{N(x,y)} dy = 0$$

By convention, we let $M(x,y)$ be the co-factor of dx and we let $N(x,y)$ be the co-factor of dy.

i.e., $M(x,y) = 16x^3y^2 + 12xy^2$ and $N(x,y) = 8yx^4 + 12yx^2 - 3e^{-y}$

If the Differential equation is **exact,** then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 32yx^3 + 24yx = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x,y)$ such that the equation $U(x,y) = C$ relates the solution y implicitly as a function of x.

To find $U(x,y)$, we compute the integrals $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$U(x,y) = \int M(x,y)\,dx = \int \left(16x^3y^2 + 12xy^2\right) dx = 4x^4y^2 + 6x^2y^2 + f(y) + C$

$U(x,y) = \int N(x,y)\,dy = \int \left(8yx^4 + 12yx^2 - 3e^{-y}\right) dy = 3e^{-y} + 6x^2y^2 + 4x^4y^2 + g(x) + C$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$$\begin{array}{ccccccccccc} U(x,y) & = & 0 & + & 4x^4y^2 & + & 6x^2y^2 & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ U(x,y) & = & g(x) & + & 4x^4y^2 & + & 6x^2y^2 & + & 3e^{-y} & + & C \end{array}$$

Thus, $f(y) = 3e^{-y}$ and $g(x) = 0$

Our solution $y = y(x)$ is given implicitly by the equation $U(x,y) = C$

$$\boxed{4x^4y^2 + 6x^2y^2 + 3e^{-y} = C}$$

6. Solve the differential equation $\frac{dy}{dx} = \frac{x+2}{y^2}$, subject to the initial condition $y(2) = 3$

Let's see if we can separate the variables.

$\frac{dy}{dx} = \frac{x+2}{y^2} \Rightarrow y^2 dy = (x+2)\,dx$ (Yipes! I didn't expect it to be THAT easy!)

Integrate:

$\int y^2 dy = \int (x+2)\,dx \Rightarrow \frac{1}{3}y^3 = \frac{1}{2}x^2 + 2x + C \Rightarrow y^3 = \frac{3}{2}x^2 + 6x + C$

Incorporating the initial condition, $y(2) = 3$, we have:

$(3)^3 = \frac{3}{2}(2)^2 + 6(2) + C \Rightarrow 27 = 18 + C$

$\Rightarrow C = 9$

Thus, $y^3 = \frac{3}{2}x^2 + 6x + 9 \Rightarrow y = \left(\frac{3}{2}x^2 + 6x + 9\right)^{\frac{1}{3}}$
Our solution y is given by:

$$\boxed{y = \left(\frac{3}{2}x^2 + 6x + 9\right)^{\frac{1}{3}}}$$

7. Solve the differential equation $y' + \cot(x)\, y = x$ (Assume that $0 < x < \frac{\pi}{2}$)

This fits the form:

$$y' + P(x)\, y = Q(x)\,, \quad \text{with } P(x) = \cot(x)\,, \text{ and } Q(x) = x$$

We can solve this using the "Integrating Factor Method"

1. Compute the integrating factor:

$e^{\int P(x)dx} = e^{\int \cot(x)dx} = e^{\ln(\sin(x))} = \sin(x)$

2. Multiply both sides by the integrating factor

$y' \sin(x) + \cot(x) \sin(x)\, y = x \sin(x)$

3. Express the left hand side as the derivative of a product

$\Rightarrow \frac{d}{dx}\left[\sin(x)\, y\right] = x \sin(x)$

4. Integrate both sides w.r.t. x.

$\int \left(\frac{d}{dx}\left[\sin(x)\, y\right]\right) dx = \int x \sin(x)\, dx$

$\Rightarrow \sin(x)\, y = \int \underbrace{x}_{u} \underbrace{\sin(x) dx}_{dv} = \underbrace{x}_{u} \underbrace{(-\cos(x))}_{v} - \int \underbrace{(-\cos(x))}_{v} \underbrace{dx}_{du} = -x\cos(x) + \sin(x) + C$

$\Rightarrow \sin(x)\, y = -x\cos(x) + \sin(x) + C \Rightarrow y = -x\cot(x) + 1 + C\csc(x)$

i.e., $y = -x\cot(x) + 1 + C\csc(x)$

Our solution y is given by the equation:

$$\boxed{y = -x\cot(x) + 1 + C\csc(x)}$$

8. Solve the differential equation $\left(xe^{\frac{y}{x}} - y\right) dx = -xdy$

We might be able to re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$\left(xe^{\frac{y}{x}} - y\right) dx = -xdy$

$\Rightarrow \left(xe^{\frac{y}{x}} - y\right) = -x\frac{dy}{dx}$

$\Rightarrow \left(e^{\frac{y}{x}} - \frac{y}{x}\right) = -\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x} - e^{\frac{y}{x}}\right)$ (i.e., $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$)

let $v = \frac{y}{x}$ (i.e., $y = vx$) $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting into the equation $\frac{dy}{dx} = \left(\frac{y}{x} - e^{\frac{y}{x}}\right)$, we have:

$v + x\frac{dv}{dx} = v - e^v$ Now Separate!

$\Rightarrow x\frac{dv}{dx} = -e^v$

$\Rightarrow -x\frac{dv}{dx} = e^v$

$e^{-v}dv = -\frac{1}{x}dx$ Now Integrate!

$\int e^{-v}dv = -\int \frac{1}{x}dx \Rightarrow -e^{-v} = -\ln|x| + C \Rightarrow e^{-v} = \ln|x| + C_1$

Now re-express in terms of x and y

$\Rightarrow e^{-\frac{y}{x}} = \ln|x| + C_1$

Our solution y is given implicitly by the equation:

$$\boxed{e^{-\frac{y}{x}} = \ln|x| + C}$$

9. Test the First-Order, Linear Differential Equation for exactness. If the equation is exact, find the solution.

$\left(4x^3y^2 + 2xy^2\right) dx + \left(8x^4 + 12x^2\right) dy = 0$

$\underbrace{\left(4x^3y^2 + 2xy^2\right)}_{M(x,y)} dx + \underbrace{\left(8x^4 + 12x^2\right)}_{N(x,y)} dy = 0$

By convention, we let $M(x, y)$ be the co-factor of dx and we let $N(x, y)$ be the co-factor of dy.

i.e., $M(x, y) = 4x^3y^2 + 2xy^2$ and $N(x, y) = 8x^4 + 12x^2$

If the Differential equation is **exact,** then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 8yx^3 + 4yx \neq 32x^3 + 24x = \frac{\partial N}{\partial x}$

The equation is NOT Exact

10. Solve the differential equation $\frac{dy}{dx} = \frac{2y+1}{x-3}$, subject to the initial condition $y(4) = 1$ (Assume that $x, y > 0$)

Let's see if we can separate the variables.

$\frac{dy}{dx} = \frac{2y+1}{x-3} \Rightarrow \frac{1}{2y+1} dy = \frac{1}{x-3} dx$ (Yipes – Another easy one!)

Integrate:

$\int \frac{1}{2y+1} dy = \int \frac{1}{x-3} dx$

Scratchwork:

$$\int \underbrace{\frac{1}{2y+1}}_{\frac{1}{u}} \underbrace{dy}_{\frac{1}{2}du} = \int \frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln(u) + C = \frac{1}{2}\ln(2y+1) + C$$

Back to our equation:

$\int \frac{1}{2y+1} dy = \int \frac{1}{x-3} dx \Rightarrow \frac{1}{2}\ln(2y+1) = \ln(x-3) + C \Rightarrow \ln(2y+1) = 2\ln(x-3) + C$

$\Rightarrow \ln(2y+1) = \ln(x-3)^2 + C \Rightarrow e^{\ln(2y+1)} = e^{\ln(x-3)^2 + C} \Rightarrow 2y+1 = e^{\ln(x-3)^2} e^C$

$\Rightarrow y + \frac{1}{2} = C_1 e^{\ln(x-3)^2} \Rightarrow y = C_1 e^{\ln(x-3)^2} - \frac{1}{2} \Rightarrow y = C_1 (x-3)^2 - \frac{1}{2}$

i.e., $y = C_1 (x-3)^2 - \frac{1}{2}$

Incorporating the initial condition, $y(4) = 1$, we have:

$(1) = C_1 (4-3)^2 - \frac{1}{2}$

$\Rightarrow C_1 = \frac{3}{2}$

Our solution y is expressed implicitly by the equation:

$$\boxed{y = \frac{3}{2}(x-3)^2 - \frac{1}{2}}$$

11. Solve the differential equation $xy' + y = x^4 + x^2$ (Assume that $x > 0$)

Rewriting the equation as:

$$y' + \frac{1}{x}y = x^3 + x,$$

This fits the form:

$$y' + P(x)\,y = Q(x)\,, \quad \text{with } P(x) = \frac{1}{x}, \text{ and } Q(x) = x^3 + x$$

We can solve this using the "Integrating Factor Method"

1. Compute the integrating factor:

$e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln((x))} = x$

2. Multiply both sides by the integrating factor

$y'x + x\frac{1}{x}y = x\left(x^3 + x\right)$

i.e., $y'x + 1y = x^4 + x^2$

3. Express the left hand side as the derivative of a product

$\frac{d}{dx}[xy] = x^4 + x^2$

4. Integrate both sides w.r.t. x

$\Rightarrow \int \left(\frac{d}{dx}[xy]\right)dx = \int \left(x^4 + x^2\right)dx$

$\Rightarrow xy = \frac{1}{5}x^5 + \frac{1}{3}x^3 + C \Rightarrow y = \frac{1}{5}x^4 + \frac{1}{3}x^2 + Cx^{-1}$

i.e., $y = \frac{1}{5}x^4 + \frac{1}{3}x^2 + Cx^{-1}$

Our solution y is given by the equation:

$$\boxed{y = \frac{1}{5}x^4 + \frac{1}{3}x^2 + Cx^{-1}}$$

12. Solve the differential equation $xydx - \left(x^2 - y^2\right)dy = 0$ (Assume that $x, y > 0$)

We might be able to re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$xydx - \left(x^2 - y^2\right)dy = 0$

$\Rightarrow xy - \left(x^2 - y^2\right)\frac{dy}{dx} = 0$

$\Rightarrow -\left(x^2 - y^2\right)\frac{dy}{dx} = -xy$

$\frac{dy}{dx} = \frac{xy}{(x^2-y^2)}$

$\frac{dy}{dx} = \frac{\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2}$ (i.e., $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$)

let $v = \frac{y}{x}$ (i.e., $y = vx$) $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting into the equation $\frac{dy}{dx} = \frac{\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2}$, we have:

$v + x\frac{dv}{dx} = \frac{v}{1-v^2}$ Now Separate!

$x\frac{dv}{dx} = \frac{v}{1-v^2} - v = \frac{v}{1-v^2} - \frac{v-v^3}{1-v^2} = \frac{v^3}{1-v^2}$

i.e., $x\frac{dv}{dx} = \frac{v^3}{1-v^2}$

$\Rightarrow \frac{1-v^2}{v^3}dv = \frac{1}{x}dx$ Now Integrate!

$\int \frac{1-v^2}{v^3}dv = \int \frac{1}{x}dx$

Scratchwork:

$\int \frac{1-v^2}{v^3}dv = \int \left(\frac{1}{v^3} - \frac{v^2}{v^3}\right)dv = \int \left(v^{-3} - \frac{1}{v}\right)dv = \frac{v^{-2}}{-2} - \ln(v) = -\frac{1}{2v^2} - \ln(v) + C$

Back to our equation:

$\int \frac{1-v^2}{v^3}dv = \int \frac{1}{x}dx$

$\Rightarrow -\frac{1}{2v^2} - \ln(v) + C = \ln(x)$

Re-express in terms of x and y.

$\Rightarrow -\frac{1}{2\left(\frac{y}{x}\right)^2} - \ln\left(\frac{y}{x}\right) + C = \ln(x)$

$\Rightarrow -\frac{1}{2\left(\frac{y}{x}\right)^2} - (\ln(y) - \ln(x)) + C = \ln(x)$

$\Rightarrow -\frac{1}{2\left(\frac{y}{x}\right)^2} - \ln(y) + \ln(x) + C = \ln(x)$

$\Rightarrow -\frac{1}{2\left(\frac{y}{x}\right)^2} - \ln(y) + C = 0$

$\Rightarrow \frac{x^2}{2y^2} + \ln(y) = C$

Our solution y is given implicitly by the equation:

$$\boxed{\frac{x^2}{2y^2} + \ln(y) = C}$$

13. Test the First-Order, Linear Differential Equation for exactness. If the equation is exact, find the solution.

$(\cos(x) + 3ye^{xy})\,dx + \left(\tan^2(y) + 3xe^{xy} + 1\right)dy = 0$

$\underbrace{(\cos(x) + 3ye^{xy})}_{M(x,y)}dx + \underbrace{\left(\tan^2(y) + 3xe^{xy} + 1\right)}_{N(x,y)}dy = 0$

By convention, we let $M(x, y)$ be the co-factor of dx and we let $N(x, y)$ be the co-factor of dy.

i.e., $M(x, y) = \cos(x) + 3ye^{xy}$ and $N(x, y) = \tan^2(y) + 3xe^{xy} + 1$

If the Differential equation is **exact,** then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 3e^{xy} + 3xye^{xy} = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x, y)$ such that the equation $U(x, y) = C$ relates the solution y implicitly as a function of x.

To find $U(x, y)$, we compute the integrals $\int M(x, y)\,dx$ and $\int N(x, y)\,dy$.

$U(x, y) = \int M(x, y)\,dx = \int (\cos(x) + 3ye^{xy})\,dx = \sin(x) + 3e^{xy} + f(y) + C$

$U(x, y) = \int N(x, y)\,dy = \int \left(\tan^2(y) + 3xe^{xy} + 1\right)dy = \tan(y) + 3e^{xy} + g(x) + C$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x, y)\,dx$ and $\int N(x, y)\,dy$.

$$\begin{array}{ccccccccc} U(x,y) & = & \sin(x) & + & 3e^{xy} & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ U(x,y) & = & g(x) & + & 3e^{xy} & + & \tan(y) & + & C \end{array}$$

Thus, $f(y) = \tan(y)$ and $g(x) = \sin(x)$

Our solution $y = y(x)$ is given implicitly by the equation $U(x, y) = C$

$$\boxed{\sin(x) + 3e^{xy} + \tan(y) = C}$$

14. Solve the differential equation $\frac{dy}{dx} = x - xy - y + 1$. (Assume that $y > 1$)

Let's see if we can separate the variables.

$\frac{dy}{dx} = x - xy - y + 1 \Rightarrow \frac{dy}{dx} = x\left(1-y\right) + \left(1-y\right) \Rightarrow \frac{dy}{dx} = \left(x+1\right)\left(1-y\right) \Rightarrow \frac{1}{1-y}dy = \left(x+1\right)dx$

Integrate:

$\int \frac{1}{1-y}dy = \int \left(x+1\right)dx \Rightarrow -\ln\left(y-1\right) = \frac{1}{2}x^2 + x + C \Rightarrow \ln\left(y-1\right) = -\frac{1}{2}x^2 - x + C_1$

$\Rightarrow e^{\ln(y-1)} = e^{-\frac{1}{2}x^2 - x + C_1} \Rightarrow y - 1 = e^{-\frac{1}{2}x^2 - x}e^{C_1} \Rightarrow y = C_2 e^{-\frac{1}{2}x^2 - x} + 1$

Our solution y is given by the equation:

$$\boxed{y = C_2 e^{-\frac{1}{2}x^2 - x} + 1}$$

15. Solve the differential equation $\cos(x)\, y' + y = 10$ Assume that $0 < x < \frac{\pi}{2}$

Rewriting the equation as:

$$y' + \sec(x)\, y = 10\sec(x),$$

This fits the form:

$$y' + P(x)\, y = Q(x), \quad \text{with } P(x) = \sec(x), \text{ and } Q(x) = 10\sec(x)$$

We can solve this using the "Integrating Factor Method"

1. Compute the integrating factor:

$e^{\int P(x)dx} = e^{\int \sec(x)dx} = e^{\ln(\sec(x)+\tan(x))} = \sec(x) + \tan(x)$

2. Multiply both sides by the integrating factor

$y'(\sec(x) + \tan(x)) + (\sec(x) + \tan(x))\sec(x)\, y = 10\sec(x)(\sec(x) + \tan(x))$

$\Rightarrow y'(\sec(x) + \tan(x)) + \left(\sec^2(x) + \sec(x)\tan(x)\right) y = \left(10\sec^2(x) + 10\sec(x)\tan(x)\right)$

3. Express the left hand side as the derivative of a product

$\Rightarrow \frac{d}{dx}\left[(\sec(x) + \tan(x))\, y\right] = \left(10\sec^2(x) + 10\sec(x)\tan(x)\right)$

4. Integrate both sides w.r.t. x

$\Rightarrow \int \left(\frac{d}{dx}\left[(\sec(x) + \tan(x))\, y\right]\right) dx = \int \left(10\sec^2(x) + 10\sec(x)\tan(x)\right) dx$

$\Rightarrow (\sec(x) + \tan(x))\, y = 10\tan(x) + 10\sec(x) + C$

i.e., $y = 10 + \frac{C}{(\sec(x)+\tan(x))} \Rightarrow y = 10 + \frac{C}{(\sec(x)+\tan(x))}\frac{(\sec(x)-\tan(x))}{\sec(x)-\tan(x)} \Rightarrow y = 10 + C(\sec(x) - \tan(x))$

i.e., $y = 10 + C(\sec(x) - \tan(x))$

Our solution y is given by the equation:

$$\boxed{y = 10 + C(\sec(x) - \tan(x))}$$

Alternative Solution appears on the next page.

Alternative Solution:

Alternatively, we can solve this equation using Separation of Variables.

Given: $\cos(x)\,y' + y = 10$ And assuming that $0 < x < \frac{\pi}{2}$,

We can rewrite the equation as:

$y' + \sec(x)\,y = 10\sec(x)$

Let's see if we can separate the variables.

$\Rightarrow y' = 10\sec(x) - \sec(x)\,y$

$\Rightarrow y' = (10-y)\sec(x)$

$\Rightarrow \frac{dy}{dx} = (10-y)\sec(x)$

$\Rightarrow \frac{1}{10-y}dy = \sec(x)\,dx$ Integrate!

$\int \frac{1}{10-y}dy = \int \sec(x)\,dx$

Scratchwork:

$$\int \underbrace{\frac{1}{10-y}}_{\frac{1}{u}} \underbrace{dy}_{-du} = \int \frac{1}{u}(-du) = -\int \frac{1}{u}du = -\ln|u| + C = -\ln|10-y| + C$$

Back to our integrals:

$\int \frac{1}{10-y}dy = \int \sec(x)\,dx$

$\Rightarrow -\ln|10-y| = \ln|\sec(x)+\tan(x)| + C$

(Since $0 < x < \frac{\pi}{2}$, we can discard absolute value bars on the right side of the equation.)

$\Rightarrow -\ln|10-y| = \ln(\sec(x)+\tan(x)) + C$

$\Rightarrow e^{-\ln|10-y|} = e^{\ln(\sec(x)+\tan(x))+C} = e^{\ln(\sec(x)+\tan(x))}e^{C} = C(\sec(x)+\tan(x))$

$\Rightarrow e^{\ln|10-y|^{-1}} = C(\sec(x)+\tan(x))$

$\Rightarrow \frac{1}{|10-y|} = C(\sec(x)+\tan(x))$

(Since C can be positive or negative, we don't need the remaining absolute value bars)

$\Rightarrow (10-y) = C_1\frac{1}{(\sec(x)+\tan(x))} = C_1\frac{1}{(\sec(x)+\tan(x))}\frac{(\sec(x)-\tan(x))}{(\sec(x)-\tan(x))} = C_1\frac{\sec(x)-\tan(x)}{\sec^2(x)-\tan^2(x)} = C_1\frac{\sec(x)-\tan(x)}{1}$

i.e., $(10-y) = C_1(\sec(x)-\tan(x))$

$\Rightarrow y - 10 = C_2(\sec(x)-\tan(x))$

Our solution y is given by the equation:

$$y = 10 + C(\sec(x)-\tan(x))$$

16. Solve the differential equation $\left(x^3+y^3\right)dx - xy^2dy = 0$; $y(1)=0$ (Assume that $x>0$)

We might be able to re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$\left(x^3+y^3\right)dx - xy^2dy = 0$

$\Rightarrow \left(x^3+y^3\right) - xy^2\frac{dy}{dx} = 0$

$\Rightarrow -xy^2\frac{dy}{dx} = -\left(x^3+y^3\right)$

$\Rightarrow \frac{dy}{dx} = \frac{\left(x^3+y^3\right)}{xy^2}$

$\Rightarrow \frac{dy}{dx} = \frac{\left(1+\frac{y^3}{x^3}\right)}{x\frac{y^2}{x^3}} = \frac{\left(1+\frac{y^3}{x^3}\right)}{\frac{y^2}{x^2}} = \frac{1+\left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$

i.e., $\frac{dy}{dx} = \frac{1+\left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$ Now Separate!

let $v = \frac{y}{x}$ (i.e., $y = vx$) $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^3}{v^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^3}{v^2} - v = \frac{1+v^3}{v^2} - v = \frac{1+v^3}{v^2} - \frac{v^3}{v^2} = \frac{1}{v^2} = v^{-2}$

i.e., $x\frac{dv}{dx} = v^{-2}$

$\Rightarrow v^2dv = \frac{1}{x}dx$ Now Integrate!

$\Rightarrow \int v^2dv = \int \frac{1}{x}dx$

$\Rightarrow \frac{1}{3}v^3 = \ln(x) + C$
Re-express in terms of x and y

$\Rightarrow \frac{1}{3}\left(\frac{y}{x}\right)^3 = \ln(x) + C$

$\Rightarrow \frac{1}{3}\frac{y^3}{x^3} = \ln(x) + C$

$\Rightarrow y^3 = 3x^3\ln(x) + C$

Our solution y is given by the equation:

$$y = \left(3x^3\ln(x) + C\right)^{\frac{1}{3}}$$