MTH 3311 Test #1

Fall 2018

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Show CLEARLY how you arrive at your answers.

1. Classify the following according to **order** and **linearity.** If an equation is **not linear,** explain why.

   (a) $y''' + x^2 y'' - 2xy = x^2 - 4$ **order 3, linear.**

   The highest order of derivative of y is 3. Furthermore, y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function, so the equation is linear.

   (b) $y^{(5)} + x^2 y y'' = \sin(x)$ **order 5, non-linear.**

   The highest order of derivative of y is 5. ($y^{(5)}$ is the *fifth derivative* of y – it is NOT y^5.) Since y'' is a "co-factor" of y, the equation is non-linear.

   (c) $y^{(4)} + 2xy'' + y^2 = 6x - 6$ **order 4, non-linear.**

   The highest order of derivative of y is 4. ($y^{(4)}$ is the *fourth derivative* of y – it is NOT y^4.) Since y is raised to a power other than 1, the equation is non-linear.

   (d) $\sin(x)\, y''' - 3xy' + x^3 y = e^x + \sin(x)$ **order 3, linear.**

   The highest order of derivative of y is 3. Furthermore, y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function, so the equation is linear.

   (e) $y'' - y' + 4y = \frac{x}{x^2+1}$ **order 2, linear.**

   The highest order of derivative of y is 2. Furthermore, y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function, so the equation is linear.

2. Show that the function $y = c_1 e^{-3x} + c_2 e^{3x} + 4x^2 + 6x$ is a solution of the differential equation $y'' - 9y = -36x^2 - 54x + 8$

   Observe:

$$\boxed{\begin{array}{lcl} y & = & c_1 e^{-3x} + c_2 e^{3x} + 4x^2 + 6x \\ y' & = & -3c_1 e^{-3x} + 3c_2 e^{3x} + 8x + 6 \\ y'' & = & 9c_1 e^{-3x} + 9c_2 e^{3x} + 8 \end{array}}$$

   Plugging into the left side of the equation, we have:

$$\boxed{\begin{array}{lcl} y'' - 9y & = & \left(9c_1 e^{-3x} + 9c_2 e^{3x} + 8\right) - 9\left(c_1 e^{-3x} + c_2 e^{3x} + 4x^2 + 6x\right) \\ & = & (9-9)\, c_1 e^{-3x} + (9-9)\, c_2 e^{3x} + \left(8 - 36x^2 - 54x\right) \\ & = & -36x^2 - 54x + 8 \end{array}}$$

   i.e., $y'' - 9y = -36x^2 - 54x + 8$

   Hence, $y = c_1 e^{-3x} + c_2 e^{3x} + 4x^2 + 6x$ is a solution of the differential equation:

   $y'' - 9y = -36x^2 - 54x + 8.$

3. Solve: $\frac{dy}{dx} = xy + 2y + x + 2$; subject to the initial condition $y(0) = 1$ (Assume that $x, y \geq 0$)

Solve the equation by "separating the variables," and then solve the equation using the "integrating factor method."

The variables can be separated.

$\Rightarrow \frac{dy}{dx} = (x+2)\,y + x + 2$

$\Rightarrow \frac{dy}{dx} = (x+2)\,y + (x+2)$

$\Rightarrow \frac{dy}{dx} = (x+2)(y+1)$

$\Rightarrow \frac{1}{y+1}dy = (x+2)\,dx$

$\Rightarrow \int \frac{1}{y+1}dy = \int (x+2)\,dx$

$\Rightarrow \ln(y+1) = \frac{1}{2}x^2 + 2x + C$

$\Rightarrow e^{\ln(y+1)} = e^{\frac{1}{2}x^2+2x+C} = e^{\frac{1}{2}x^2+2x}e^{C} = e^{\frac{1}{2}x^2+2x}C_1 = C_1e^{\frac{1}{2}x^2+2x}$

i.e., $y + 1 = C_1e^{\frac{1}{2}x^2+2x}$

$y = C_1e^{\frac{1}{2}x^2+2x} - 1$

Recall: $y(0) = 1$

$\Rightarrow 1 = C_1e^{\frac{1}{2}(0)^2+2(0)} - 1 = C_1 - 1$

$\Rightarrow 2 = C_1$

$$\boxed{\Rightarrow y = 2e^{\frac{1}{2}x^2+2x} - 1}$$

Alternative Solution on the next page

Alternatively: We can solve this equation using the "Integrating Factor Method"

i) Re-express the equation in the form: $y' + p(x)\,y = Q(x)$

$\frac{dy}{dx} = xy + 2y + x + 2$

$\Rightarrow y' = xy + 2y + x + 2$

$\Rightarrow y' = (x+2)\,y + x + 2$

$\Rightarrow y' - (x+2)\,y = (x+2)$

$\Rightarrow y' + \underbrace{(-x-2)}_{p(x)}y = (x+2)$

ii) Compute the integrating factor: $e^{\int p(x)dx} = e^{\int(-x-2)dx} = e^{-\frac{1}{2}x^2-2x}$

iii) Multiply both sides by the integrating factor

$\Rightarrow e^{-\frac{1}{2}x^2-2x}y' + (-x-2)\,y = (x+2)\,e^{-\frac{1}{2}x^2-2x}$

iv) Express the left side as the derivative of a product

$\Rightarrow \left[e^{-\frac{1}{2}x^2-2x}y\right] = (x+2)\,e^{-\frac{1}{2}x^2-2x}$

v) Integrate!

$\Rightarrow \int \frac{d}{dx}\left[e^{-\frac{1}{2}x^2-2x}y\right]dx = \int (x+2)\,e^{-\frac{1}{2}x^2-2x}dx$

i.e., $\int \frac{d}{dx}\left[e^{-\frac{1}{2}x^2-2x}y\right]dx = \int \underbrace{e^{-\frac{1}{2}x^2-2x}}_{e^u}\underbrace{(x+2)dx}_{-du} = \int e^u(-du) = -\int e^u du = -e^u + C = -e^{-\frac{1}{2}x^2-2x} + C$

$\Rightarrow \left[e^{-\frac{1}{2}x^2-2x}y\right] = -e^{-\frac{1}{2}x^2-2x} + C$

vi) Solve for y

$\Rightarrow y = -1 + Ce^{\frac{1}{2}x^2+2x}$

i.e., $y = Ce^{\frac{1}{2}x^2+2x} - 1$

Incorporating the initial condition $y(0) = 1$, we have:

$\Rightarrow 1 = Ce^{\frac{1}{2}(0)^2+2(0)} - 1 = C - 1$

i.e., $1 = C - 1$

$\Rightarrow C = 2$

$$\boxed{\Rightarrow y = 2e^{\frac{1}{2}x^2+2x} - 1}$$

4. Solve: $y' + \frac{1}{x+1}y = \frac{1}{x+1}\sin(x)$; using the "integrating factor method." (Assume that $x, y > 0$.)

i) Express the equation in the form: $y' + p(x)\,y = Q(x)$

$$y' + \underbrace{\frac{1}{x+1}}_{p(x)}y = \underbrace{\frac{1}{x+1}\sin(x)}_{Q(x)}$$

ii) Compute the integrating factor: $e^{\int p(x)dx} = e^{\int\left(\frac{1}{x+1}\right)dx} = \underbrace{e^{\ln|x+1|} = e^{\ln(x+1)}}_{\text{because } x>0} = (x+1)$

iii) Multiply both sides by the integrating factor

$\Rightarrow (x+1)\,y' + (x+1)\,\frac{1}{x+1}y = (x+1)\,\frac{1}{x+1}\sin(x)$

i.e., $(x+1)\,y' + y = \sin(x)$

iv) Express the left side as the derivative of a product

$\Rightarrow \frac{d}{dx}\left[(x+1)\,y\right] = \sin(x)$

v) Integrate!

$\Rightarrow \int \frac{d}{dx}\left[(x+1)\,y\right]dx = \int \sin(x)\,dx$

i.e., $(x+1)\,y = -\cos(x) + C$

vi) Solve for y

$$\boxed{y = -\frac{\cos(x)}{x+1} + \frac{C}{x+1} = -\frac{\cos(x)+C_1}{x+1}}$$

5. Determine whether or not the equation is exact. If the equation is exact, solve it.

$$\underbrace{\left(3x^2y + y\cos(x) + 2y^3 + e^x\right)}_{M}dx + \underbrace{\left(x^3 + 6xy^2 + \sin(x) + \cos(y)\,e^{\sin(y)}\right)}_{N}dy = 0$$

The equation will be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check:

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[3x^2y + y\cos(x) + 2y^3 + e^x\right] = \cos(x) + 3x^2 + 6y^2$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[x^3 + 6xy^2 + \sin(x) + \cos(y)\,e^{\sin(y)}\right] = \cos(x) + 3x^2 + 6y^2$

i.e., $\frac{\partial M}{\partial y} = \cos(x) + 3x^2 + 6y^2 = \frac{\partial N}{\partial x}$

Hence, the original equation is exact.

The solution to the Differential Equation is of the form: $U(x,y) = C$,

where $U(x,y) = \int M dx = \int N dy$.

$U(x,y) = \int M dx = \int\left[3x^2y + y\cos(x) + 2y^3 + e^x\right]dx = x^3y + y\sin(x) + 2xy^3 + e^x + f(y) = C$

$U(x,y) = \int N dx = \int\left[x^3 + 6xy^2 + \sin(x) + \cos(y)\,e^{\sin(y)}\right]dx = x^3y + 2xy^3 + y\sin(x) + e^{\sin(y)} + g(x) = C$

Comparing terms, we see that $f(y) = e^{\sin(y)}$ and $g(x) = e^x$

Thus, $U = x^3y + 2xy^3 + \sin(x)\,y + e^x + e^{\sin(y)} = C$

Our solution y is given implicitly by the equation:

$$\boxed{x^3y + 2xy^3 + \sin(x)\,y + e^x + e^{\sin(y)} = C}$$

6. Solve: $2xy\frac{dy}{dx} = 3x^2 + 4y^2$ using the substitution $v = \frac{y}{x}$. (Assume that $x, y > 0$)

i) Re-write in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$2xy\frac{dy}{dx} = 3x^2 + 4y^2$

$\Rightarrow 2\frac{y}{x}\frac{dy}{dx} = 3 + 4\left(\frac{y}{x}\right)^2$

$$\Rightarrow \frac{dy}{dx} = \underbrace{\frac{3}{2}\frac{1}{\left(\frac{y}{x}\right)} + 2\left(\frac{y}{x}\right)}_{f\left(\frac{y}{x}\right)}$$

ii) Make the following substitutions: $v = \frac{y}{x}$; $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{3}{2}\frac{1}{v} + 2v$

$\Rightarrow x\frac{dv}{dx} = \frac{3}{2}\frac{1}{v} + v = \frac{3}{2v} + v = \frac{3}{2v} + \frac{2v^2}{2v} = \frac{2v^2+3}{2v}$

i.e. $x\frac{dv}{dx} = \frac{2v^2+3}{2v}$

iii) Separate!

$\frac{2v}{2v^2+3}dv = \frac{1}{x}dx$

iv) Integrate:

$\int \frac{2v}{2v^2+3}dv = \int \frac{1}{x}dx$ (Eq. 1)

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Scratchwork:

$$\int \frac{2v}{2v^2+3}dv = \int \underbrace{\frac{1}{2v^2+3}}_{\frac{1}{u}}\underbrace{2vdv}_{\frac{1}{2}du} = \int \frac{1}{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln\left|2v^2+3\right| = \frac{1}{2}\ln\left(2v^2+3\right)$$

i.e., $\int \frac{2v}{2v^2+3}dv = \frac{1}{2}\ln\left(2v^2+3\right)$

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Substituting this into Eq. 1, we have:

$\frac{1}{2}\ln\left(2v^2+3\right) = \ln(x) + C$

$\Rightarrow \ln\left(2v^2+3\right) = 2\ln(x) + C_1$

$\Rightarrow \ln\left(2v^2+3\right) = \ln\left(x^2\right) + C_1$

$\Rightarrow e^{\ln\left(2v^2+3\right)} = e^{\ln\left(x^2\right)+C_1} = e^{\ln\left(x^2\right)}e^{C_1} = C_2e^{\ln\left(x^2\right)} = C_2x^2$

i.e., $2v^2 + 3 = C_2x^2$

$\Rightarrow 2\left(\frac{y}{x}\right)^2 + 3 = C_2x^2$

$\Rightarrow 2\left(\frac{y^2}{x^2}\right) + 3 = C_2x^2$

$\Rightarrow 2y^2 + 3x^2 = C_2 x^4$

Our solution is given implicitly by the equation:

$$\boxed{2y^2 + 3x^2 = C_2 x^4}$$