Differential Equations Test #1

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Instructions. Show clearly how you arrive at your answers.

1. Classify the following according to **order** and **linearity.**

   (a) $y^{(4)} - xy'' - 5x^2y' + 6y = e^x$ **order 4, linear.**

   The highest order of derivative of y is 4. ($y^{(4)}$ is the *fourth derivative* of y – it is NOT y^4.) Since the degree of y and its derivatives is one in each term, and since y and its derivatives are never the inner function of a composite function, the equation is linear.

   (b) $y' = \frac{x}{y}$ **order 1, non-linear.**

   The highest order of the derivative of y is 1. No analysis of this equation will permit us to analyze it in such a fashion that the degree of y and its derivatives is one in each term and y and its derivatives are never the inner function of a composite function. (e.g., $y' = \frac{x}{y}$ is the same as $y' = x\frac{1}{y}$. Here, y is the inner function of the composite function $f(y)$, where $f(\ ) = \frac{1}{(\ )}$.

   Alternatively, $y' = \frac{x}{y}$ is the same as $y' = xy^{-1}$. Hence, the function y has degree -1 (when considering y and its derivatives.) Therefore, the equation is non-linear.

   Alternatively, $y' = \frac{x}{y}$ is the same as $yy' = x$. Here, the term yy' has degree two (when considering y and its derivatives.)
   Therefore, the equation is non-linear.

   (c) $\frac{d^2s}{dt^2} = -9s^2$ **order 2, non-linear.**

   The highest order of derivative of s is 2. The term $-9s^2$ has degree two (when considering s and its derivatives.)
   Therefore, the equation is non-linear.

   (d) $3y'' - y' - 10xy = 10$ **order 2, linear.**

   The highest order of derivative of y is 2. The degree of y and its derivatives is one in each term, and y and its derivatives are never the inner function of a composite function. The equation is linear.

2. Solve: $\frac{dy}{dx} = -xy; \quad y = 5$ when $x = 0$

By Separation of Variables, we have:

$\frac{dy}{dx} = -xy \Rightarrow \frac{1}{y}dy = -xdx \Rightarrow \int \frac{1}{y}dy = -\int xdx \Rightarrow \ln(y) = -\frac{x^2}{2} + C$

$$\Rightarrow e^{\ln(y)} = e^{-\frac{x^2}{2}+C} \Rightarrow y = e^{-\frac{x^2}{2}} \cdot e^C \Rightarrow y = C_1 e^{-\frac{x^2}{2}}$$

i.e., the *general solution* is $y = C_1 e^{-\frac{x^2}{2}}$.

Now, we'll use the initial conditions to find C_1.

Recall: $y = 5$ when $x = 0$

$\Rightarrow 5 = C_1 e^{-\frac{(0)^2}{2}} \Rightarrow 5 = C_1$.

$y = 5e^{-\frac{x^2}{2}}$ is the particular solution.

3. Show that the function $y = c_1 e^{-2x} + c_2 e^{3x} + x$ is a solution of the differential equation $y'' - y' - 6y = -6x - 1$.

Observe:

$$\begin{aligned} y &= c_1 e^{-2x} + c_2 e^{3x} + x \\ y' &= -2c_1 e^{-2x} + 3c_2 e^{3x} + 1 \\ y'' &= 4c_1 e^{-2x} + 9c_2 e^{3x} \end{aligned}$$

Plugging into the left side of the equation, we have:

$$\begin{aligned} y'' - y' - 6y &= \left(4c_1 e^{-2x} + 9c_2 e^{3x}\right) - \left(-2c_1 e^{-2x} + 3c_2 e^{3x} + 1\right) - 6\left(c_1 e^{-2x} + c_2 e^{3x} + x\right) \\ &= \left(4 - (-2) - 6\right) c_1 e^{-2x} + \left(9 - 3 - 6\right) c_2 e^{3x} + \left(-1 - 6x\right) \\ &= -6x - 1 \end{aligned}$$

i.e., $y'' - y' - 6y = -6x - 1$

Hence, $y = c_1 e^{-2x} + c_2 e^{3x} + x$ is a solution of the differential equation:

$y'' - y' - 6y = -6x - 1$.

4. Solve: $\frac{dy}{dx} - y = x^2 y \quad y(0) = 1$. (Assume that $x, y > 0$)

We can separate this one. As we do, we'll assume that $y > 0$.

$\frac{dy}{dx} - y = x^2 y \Rightarrow \frac{dy}{dx} = x^2 y + y \Rightarrow \frac{dy}{dx} = y\left(x^2+1\right)$

$\Rightarrow \frac{1}{y} dy = \left(x^2+1\right) dx$

$\Rightarrow \int \frac{1}{y} dy = \int \left(x^2+1\right) dx \Rightarrow \ln(y) = \frac{1}{3}x^3 + x + C_1$

$\Rightarrow e^{\ln(y)} = e^{\frac{1}{3}x^3 + x + C_1} \Rightarrow y = e^{C_1} e^{\frac{1}{3}x^3 + x} \Rightarrow y = C_2 e^{\frac{1}{3}x^3 + x}$

This is the *general solution.* To find the *particular solution,* we need to evaluate the arbitrary constant of integration, C_2.

Our initial condition is $y = 1$ when $x = 0$. Plugging into the general solution, we have:

$1 = C_2 e^{\frac{1}{3}(0)^3 + (0)} \Rightarrow 1 = C_2$.

Our particular solution is $y = e^{\frac{1}{3}x^3 + x}$

5. Solve: $\frac{1}{x}y' + 2y = x^2 + \frac{1}{x}$

   If we multiply both sides by x, the equation becomes:

   $y' + 2xy = x^3 + 1$

   Our best strategy appears to be to get this equation in the "First Order, Linear" form, $y' + P(x)\,y = Q(x)$.

   (a) 1. $y' + \underbrace{2x}_{P(x)}y = \underbrace{x^3 + 1}_{Q(x)}$

   2. Next, Multiply by our integrating factor, $e^{\int P(x)dx}$. $e^{\int P(x)dx} = e^{\int 2xdx} = e^{x^2}$

      Multiplying both sides by the integrating factor, we have:

      $\Rightarrow e^{x^2}\left(y' + 2xy\right) = e^{x^2}\left(x^3 + 1\right)$

   3. Express the L.H. side as the derivative of a product (The integrating factor times y)

      $\Rightarrow \frac{d}{dx}\left[e^{x^2}y\right] = e^{x^2}\left(x^3 + 1\right)dx$

   4. Integrate:

      $\Rightarrow \int \frac{d}{dx}\left[e^{x^2}y\right]dx = \int e^{x^2}\left(x^3 + 1\right)dx$

      $\Rightarrow \left[e^{x^2}y\right] = \int e^{x^2}\left(x^3 + 1\right)dx$

      We can't compute the integral on the R.H. side

   5. Divide both sides by the cofactor of y.

      Essentially our solution is: $y = \frac{\int e^{x^2}\left(x^3+1\right)dx}{e^{x^2}}$

6. Show that the equation is exact, and solve: $(6x - 3y)\,dx + (4y - 3x)\,dy = 0$

This has the form: $\underbrace{(6x - 3y)}_{M}dx + \underbrace{(4y - 3x)}_{N}dy = 0$

The equation will be exact, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check:

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(6x - 3y) = -3$
$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(4y - 3x) = -3$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation IS exact.

Our solution y can be expressed explicitly by the equation $U(x, y) = C$, where U has the characteristics that

$$M = \frac{\partial U}{\partial x} \quad \text{and} \quad N = \frac{\partial U}{\partial y}$$

$\Rightarrow U = \int M \partial x = \int (6x - 3y)\,\partial x = 3x^2 - 3yx + F(y)$

$\Rightarrow U = \int N \partial y = \int (4y - 3x)\,\partial y = 2y^2 - 3xy + G(x)$

Comparing the two, we have:

$$3x^2 \quad - \quad 3yx \quad + \quad F(y) \quad = \quad 2y^2 \quad - \quad 3xy \quad + \quad G(x)$$

From our comparison, we see that $F(y) = 2y^2$ and $G(x) = 3x^2$.

Hence, our solution y is expressed implicitly by the equation:

$U(x, y) = 3x^2 - 3xy + 2y^2 = C$

7. Solve: $y' = \frac{x-y}{x+y}$; using the substitution $v = \frac{y}{x}$

Our immediate goal is to re-write this equation in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

$y' = \frac{x-y}{x+y} \Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y} \Rightarrow \frac{dy}{dx} = \frac{1-\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)}$.

Using the substitution, $v = \frac{y}{x}$, we have $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

Substituting these into the equation, $\frac{dy}{dx} = \frac{1-\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)}$, we have:

$x\frac{dv}{dx} + v = \frac{1-v}{1+v}$

Now separate the variables!

$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v \Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - \frac{1+v}{1+v}v \Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - \frac{v+v^2}{1+v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v} \Rightarrow \frac{v+1}{v^2+2v-1}dv = -\frac{1}{x}dx$

Integrate!

$$\int \frac{v+1}{v^2+2v-1}dv = \int \underbrace{\frac{1}{v^2+2v-1}}_{\frac{1}{u}} \underbrace{(v+1)\,dv}_{\frac{1}{2}du} = \int \frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}du$$

$$= \frac{1}{2}\ln(u) + C = \frac{1}{2}\ln\left(v^2+2v-1\right) + C$$

Thus, the equation $\frac{v+1}{v^2+2v-1}dv = -\frac{1}{x}dx$ becomes:

$\int \frac{v+1}{v^2+2v-1}dv = -\int \frac{1}{x}dx$

$\Rightarrow \frac{1}{2}\ln\left(v^2+2v-1\right) = -\ln(x) + C$

$\Rightarrow \ln\left(v^2+2v-1\right) = -2\ln(x) + C$

$\Rightarrow \ln\left(v^2+2v-1\right) = \ln\left(x^{-2}\right) + C$

$\Rightarrow e^{\ln\left(v^2+2v-1\right)} = e^{\ln\left(x^{-2}\right)+C}$

$\Rightarrow v^2+2v-1 = C_1 e^{\ln\left(x^{-2}\right)}$

$\Rightarrow v^2+2v-1 = C_1 x^{-2}$

$\Rightarrow \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) - 1 = C_1 x^{-2}$

$\Rightarrow \frac{y^2}{x^2} + 2\frac{y}{x} - 1 = C_1 x^{-2}$

$\Rightarrow y^2 + 2xy - x^2 = C_1$ general solution