MTH 3311 Differential Equations Test #1 - Solutions

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1. Classify the following according to **order** and **linearity.**

   (a) $y^{(4)} - x^2 y'' - 5xy' + 6y = e^x$ **order 4, linear.**

   The highest order of derivative of y is 4. ($y^{(4)}$ is the *fourth derivative* of y – it is NOT y^4.) Since neither y nor its derivatives: [1] is raised to a power other than 1; [2] appear as the "inner function" of a composite function; and [3] are not the "co-factors" of y, or one of its derivatives, the equation is linear.

   (b) $(y')^2 = 2xy$ **order 1, non-linear.**

   The highest order of derivative of y is 1. (y' is the *first derivative* of y), so the order is 1. Since y' is raised to a power other than 1, the equation is non-linear.

   (c) $\frac{d^2y}{dx^2} + \sin(x) \frac{dy}{dx} = \frac{9x^2+3x}{x^2+1}$ **order 2, linear.**

   The highest order of derivative of y is 2. ($\frac{d^2y}{dx^2}$ is the *second derivative* of y.) Since neither y nor its derivatives: [1] is raised to a power other than 1; [2] appear as the "inner function" of a composite function; and [3] are not the "co-factors" of y, or one of its derivatives, the equation is linear.

   (d) $y''' - 5xy' - 10xy^3 = 10$ **order 3, non-linear.**

   The highest order of derivative of y is 3. (y''' is the *third derivative* of y.) Since y is raised to the third power in the term $10xy^3$, the equation is non-linear.

   (e) $6y^{(10)} - y^{(5)} - 10xy' = \tan(x)$ **order 2, linear.**

   The highest order of derivative of y is 10. ($y^{(10)}$ is the *tenth derivative* of y.) Since neither y nor its derivatives: [1] is raised to a power other than 1; [2] appear as the "inner function" of a composite function; and [3] are not the "co-factors" of y, or one of its derivatives, the equation is linear.

2. Show that the function $y = c_1 e^{-3x} + c_2 e^x + 3x^2 + 5$ is a solution of the differential equation $y'' + 2y' - 3y = -9x^2 + 12x - 9$

   Observe:

$$\boxed{\begin{array}{lcl} y & = & c_1 e^{-3x} + c_2 e^x + 3x^2 + 5 \\ y' & = & -3c_1 e^{-3x} + c_2 e^x + 6x \\ y'' & = & 9c_1 e^{-3x} + c_2 e^x + 6 \end{array}}$$

   Plugging into the left side of the equation, we have:

$$\boxed{\begin{array}{lcl} y'' + 2y' - 3y & = & \left(9c_1 e^{-3x} + c_2 e^x + 6\right) + 2\left(-3c_1 e^{-3x} + c_2 e^x + 6x\right) - 3\left(c_1 e^{-3x} + c_2 e^x + 3x^2 + 5\right) \\ & = & 9c_1 e^{-3x} + c_2 e^x + 6 - 6c_1 e^{-3x} + 2c_2 e^x + 12x - 3c_1 e^{-3x} - 3c_2 e^x - 9x^2 - 15 \\ & = & (9 - 6 - 3)\, c_1 e^{-3x} + (1 + 2 - 3)\, c_2 e^{2x} + 6 + 12x - 9x^2 - 15 \\ & = & -9x^2 + 12x - 9 \end{array}}$$

   i.e., $y'' + y' - 6y = -9x^2 + 12x - 9$

   Hence, $y = c_1 e^{-3x} + c_2 e^x + 3x^2 + 5$ is a solution of the differential equation:

   $y'' + 2y' - 3y = -9x^2 + 12x - 9.$

3. Solve: $\frac{dy}{dx} = x^2y - x^2$; Subject to the initial condition: $y(0) = 3$.

(Assume that $x \geq 0$ and $y \geq 2$).

We can separate this one.

$\frac{dy}{dx} = x^2y - x^2 \Rightarrow \frac{dy}{dx} = x^2(y-1) \Rightarrow dy = x^2(y-1)\,dx \Rightarrow \frac{1}{y-1}dy = x^2dx$

Integrate!

$\int \frac{1}{y-1}dy = \int x^2dx \Rightarrow \ln(y-1) = \frac{1}{3}x^3 + C$

$\Rightarrow e^{\ln(y-1)} = e^{\frac{1}{3}x^3+C} \Rightarrow y - 1 = e^Ce^{\frac{1}{3}x^3} \Rightarrow y - 1 = C_1e^{\frac{1}{3}x^3} \Rightarrow y = C_1e^{\frac{1}{3}x^3} + 1$ (Eq. 1)

This is the *general solution.* To find the *particular solution,* we need to evaluate the arbitrary constant of integration, C_1.

Our initial condition is $y = 3$ when $x = 0$. Plugging into the general solution, we have:

$3 = y = C_1e^{\frac{1}{3}(0)^3} + 1 \Rightarrow 3 = C_1 + 1 \Rightarrow C_1 = 2$

Plugging $C_1 = 2$ into Eq. 1, we have: $y = 2e^{\frac{1}{3}x^3} + 1$

Our particular solution is $y = 2e^{\frac{1}{3}x^3} + 1$

Alternatively: The equation can be solved using the "Integrating Factor Method"

(Next Page)

Alternatively: The equation $\frac{dy}{dx} = x^2y - x^2$; Subject to the initial condition: $y(0) = 3$, can be solved using the "Integrating Factor Method."

$$\frac{dy}{dx} = x^2y - x^2 \Rightarrow \frac{dy}{dx} - x^2y = -x^2 \Rightarrow \frac{dy}{dx} + \underbrace{\left(-x^2\right)}_{P(x)} y = \underbrace{-x^2}_{Q(x)}$$

i. Compute the integrating factor:

Our integrating factor is $e^{\int P(x)dx} = e^{\int -x^2 dx} = e^{-\frac{1}{3}x^3}$

ii. Multiply both sides by the integrating factor

$$\Rightarrow e^{-\frac{1}{3}x^3}\left[\frac{dy}{dx} - x^2y\right] = -x^2e^{-\frac{1}{3}x^3} \Rightarrow e^{-\frac{1}{3}x^3}\frac{dy}{dx} - x^2e^{-\frac{1}{3}x^3}y = -x^2e^{-\frac{1}{3}x^3}$$

iii. Express the Left hand side as the derivative of a product

$$\Rightarrow \frac{d}{dx}\left[e^{-\frac{1}{3}x^3}y\right] = -x^2e^{-\frac{1}{3}x^3}$$

iv. Integrate both sides w.r.t. x.

$$\int\left(\frac{d}{dx}\left[e^{-\frac{1}{3}x^3}y\right]\right)dx = \int\left(-x^2e^{-\frac{1}{3}x^3}\right)dx \Rightarrow e^{-\frac{1}{3}x^3}y = e^{-\frac{1}{3}x^3} + C$$

v. Solve for y

$$\Rightarrow y = 1 + e^{\frac{1}{3}x^3}C$$

This is the *general solution.* To find the *particular solution,* we need to evaluate the arbitrary constant of integration, C_1.

Our initial condition is $y = 3$ when $x = 0$. Plugging into the general solution, we have:

$$3 = y = 1 + Ce^{\frac{1}{3}(0)^3} \Rightarrow 3 = 1 + C \Rightarrow C = 2$$

Plugging $C = 2$ into Eq. 1, we have: $y = 1 + 2e^{\frac{1}{3}x^3}$

Our particular solution is $y = 1 + 2e^{\frac{1}{3}x^3}$

4. Solve: $x^4 \frac{dy}{dx} + 3x^3 y = x \tan(x)$; using the "integrating factor method."

(Assume that $0 < x < \frac{\pi}{2}$).

First, we get the equation in the form: $\frac{dy}{dx} + P(x)\, y = Q(x)$

$x^4 \frac{dy}{dx} + 3x^3 y = x \tan(x) \Rightarrow \frac{dy}{dx} + \frac{3}{x} y = \frac{\tan(x)}{x^3}$

i.e., $\frac{dy}{dx} + \underbrace{\frac{3}{x}}_{P(x)} y = \underbrace{\frac{\tan(x)}{x^3}}_{Q(x)}$

i. Compute the integrating factor:

$e^{\int P(x)dx} = e^{\int \frac{3}{x} dx} = e^{3 \int \frac{1}{x} dx} = e^{3 \ln(x)} = e^{\ln\left(x^3\right)} = x^3$

i.e., our integrating factor is x^3

ii. Multiply both sides by the integrating factor

$\Rightarrow x^3 \frac{dy}{dx} + x^3 \left(\frac{3}{x}\right) y = x^3 \left(\frac{\tan(x)}{x^3}\right)$

$\Rightarrow x^3 \frac{dy}{dx} + 3x^2 y = \tan(x)$

iii. Express the Left hand side as the derivative of a product

$\Rightarrow \frac{d}{dx}\left[x^3 y\right] = \tan(x)$

4. Integrate both sides w.r.t. x.

$\int \left(\frac{d}{dx}\left[x^3 y\right]\right) dx = \int \tan(x)\, dx \Rightarrow x^3 y = \ln(\sec(x)) + C$

Solve for y

$\Rightarrow y = x^{-3} (\ln(\sec(x)) + C)$

Thus, our solution is $y = x^{-3} (\ln(\sec(x)) + C)$

5. Determine whether or not the equation is exact. If the equation is exact, solve it.

$\left(e^y \sec^2(x) - \sin(x) + 10x\sin(y)\right)dx + \left(5x^2\cos(y) + e^y\tan(x) + \frac{1}{y}\right)dy = 0$

$$\underbrace{\left(e^y \sec^2(x) - \sin(x) + 10x\sin(y)\right)}_{M(x,y)}dx + \underbrace{\left(5x^2\cos(y) + e^y\tan(x) + \frac{1}{y}\right)}_{N(x,y)}dy = 0$$

By convention, we let $M(x,y)$ be the co-factor of dx and we let $N(x,y)$ be the co-factor of dy.

i.e., $M(x,y) = e^y\sec^2(x) - \sin(x) + 10x\sin(y)$ and $N(x,y) = 5x^2\cos(y) + e^y\tan(x) + \frac{1}{y}$

If the Differential equation is **exact,** then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = e^y\sec^2(x) + 10x\cos(y) = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x,y)$ such that the equation $U(x,y) = C$ relates the solution y implicitly as a function of x.

To find $U(x,y)$, we compute the integrals $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$U(x,y) = \int M(x,y)\,dx = \int \left(e^y\sec^2(x) - \sin(x) + 10x\sin(y)\right)dx = e^y\tan(x) + \cos(x) + 5x^2\sin(y) + f(y) + C$

$U(x,y) = \int N(x,y)\,dy = \int \left(5x^2\cos(y) + e^y\tan(x) + \frac{1}{y}\right)dy = 5x^2\sin(y) + e^y\tan(x) + \ln(y) + g(x) + C$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$$\begin{array}{ccccccccccc} U(x,y) & = & e^y\tan(x) & + & \cos(x) & + & 5x^2\sin(y) & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ U(x,y) & = & e^y\tan(x) & + & g(x) & + & 5x^2\sin(y) & + & \ln(y) & + & C \end{array}$$

Thus, $f(y) = \ln(y)$ and $g(x) = \cos(x)$, and $U(x,y) = e^y\tan(x) + \cos(x) + 5x^2\sin(y) + \ln(y) + C$

Our solution $y = y(x)$ is given (implicitly) by the equation $U(x,y) = C$

$$\boxed{e^y\tan(x) + \cos(x) + 5x^2\sin(y) + \ln(y) = C}$$

6. Solve: $2xy\frac{dy}{dx} + \left(x^2 - y^2\right) = 0$ Solve, using the substitution $v = \frac{y}{x}$

Re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$\Rightarrow 2xy\frac{dy}{dx} = y^2 - x^2$

$\Rightarrow xy\frac{dy}{dx} = \frac{1}{2}\left(y^2 - x^2\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{y^2}{xy} - \frac{x^2}{xy}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\left(\frac{y}{x}\right) - \frac{1}{\left(\frac{y}{x}\right)}\right)$

Thus, we have re-expressed the equation in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

let $v = \frac{y}{x}$ (i.e., $y = vx$) $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting, we have:

$v + x\frac{dv}{dx} = \frac{1}{2}\left(v - \frac{1}{v}\right)$ Now Separate!

$\Rightarrow x\frac{dv}{dx} = -\frac{1}{2}\left(v + \frac{1}{v}\right) \Rightarrow x\frac{dv}{dx} = -\frac{1}{2}\frac{v^2+1}{v} \Rightarrow \frac{v}{v^2+1}dv = -\frac{1}{2}\left(\frac{1}{x}dx\right)$ Now Integrate!

$\Rightarrow \int \frac{v}{v^2+1}dv = -\frac{1}{2}\int\left(\frac{1}{x}dx\right) \Rightarrow \frac{1}{2}\ln\left(v^2+1\right) = -\frac{1}{2}\ln(x) + C \Rightarrow \ln\left(v^2+1\right) = -\ln(x) + C$

$\Rightarrow \ln\left(v^2+1\right) = \ln\left(\frac{1}{x}\right) + C \Rightarrow e^{\ln\left(v^2+1\right)} = e^{\ln\left(\frac{1}{x}\right)+C} \Rightarrow v^2 + 1 = e^{\ln\left(\frac{1}{x}\right)}e^C$

$\Rightarrow v^2 + 1 = C\frac{1}{x} \Rightarrow \left(\frac{y}{x}\right)^2 + 1 = C\frac{1}{x} \Rightarrow \frac{y^2}{x^2} + 1 = C\frac{1}{x} \Rightarrow y^2 + x^2 = Cx$

Our solution y is given (implicitly) by the equation:

$$\boxed{y^2 + x^2 = Cx}$$