MTH 3311 - Test 2A - Solutions

Fall 2018

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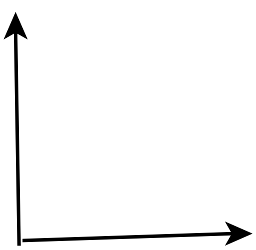
1. The force of water resistance acting on a boat is proportional to its instantaneous velocity, and is such that at 40 $\frac{\text{ft}}{\text{sec}}$ the water resistance is 80 lb. If the boat and passengers combined weigh 640 lb, and if the motor exerts a steady force of 100 lb in the direction of the motion:

   (a) Find the velocity at any time $t \geq 0$, assuming that the boat starts from rest.

   (b) Find the limiting velocity

First of all, we will establish positive direction(s)

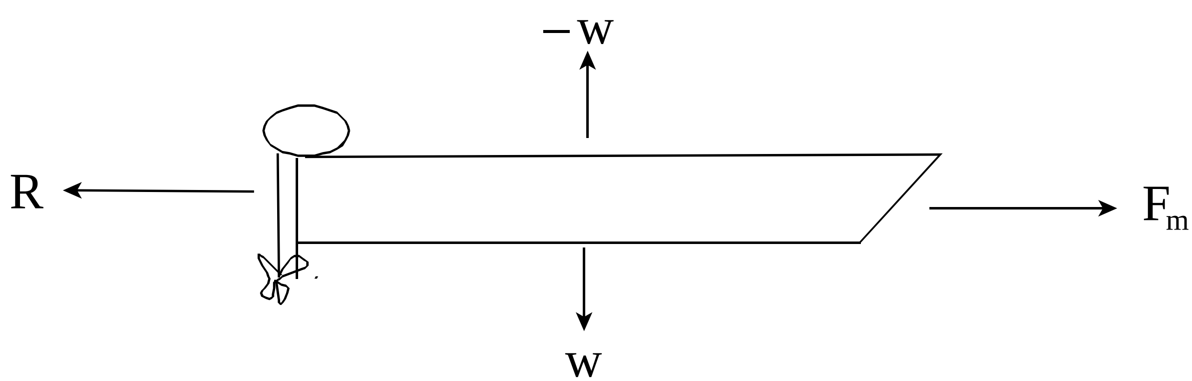


Positive Direction

Note: "The force of water resistance R is proportional to its instantaneous velocity v."

i.e., $R = kv$, where k is the constant of proportionality.

We draw a "force diagram" of the boat (below).



The total force acting on the boat *(in the horizontal direction)* is given by:

$F = R + F_m$, where F_m is the force applied by the motor.

We will let w be the combined weight of the boat and the passenger. The buoyant force acting on the boat is equal to $-w$.

(Note that the sum of the *vertical* forces acting on the boat is zero. Otherwise, the boat would either move upward or downward, depending on whether the sum of the vertical forces is positive or negative.)

The total (horizontal) force F is given by:

$F = R + F_m$

Here's the key to solving this type of exercise:

Recall that $F = ma$

By substituting ma for F, we get an equation in terms of velocity and acceleration:

$$\underbrace{ma}_{F} = \underbrace{kv}_{R} + \underbrace{100 \text{ lb}}_{F_m}$$

The importance of getting an equation in terms of velocity and acceleration, is that acceleration is the derivative of velocity (i.e., $a = \frac{dv}{dt}$).

Thus, by substituting $\frac{dv}{dt}$ for a, our equation becomes:

$m\frac{dv}{dt} = kv + 100$ lb., which is a differential equation in terms of velocity v.

We can solve the equation using the "Integrating Factor Method."

$\Rightarrow m\frac{dv}{dt} - kv = 100 \text{ lb}$

$\Rightarrow \frac{dv}{dt} - \frac{k}{m}v = \frac{100 \text{ lb}}{m}$

$$\Rightarrow \frac{dv}{dt} + \underbrace{\left(-\frac{k}{m}\right)}_{P(t)} v = \frac{100 \text{ lb}}{m}$$

Our integrating factor is $e^{\int p(t)dt} = e^{\int \left(-\frac{k}{m}\right)dt} = e^{-\frac{k}{m}t}$

Multiply both sides by the integrating factor

$\Rightarrow e^{-\frac{k}{m}t}\frac{dv}{dt} + \left(-\frac{k}{m}\right)e^{-\frac{k}{m}t}v = \frac{100 \text{ lb}}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right] = \frac{100 \text{ lb}}{m}e^{-\frac{k}{m}t}$

Integrate both sides with respect to t

$\Rightarrow \int \frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right]dt = \frac{100 \text{ lb}}{m}\int e^{-\frac{k}{m}t}dt$

$\Rightarrow e^{-\frac{k}{m}t}v = \frac{100 \text{ lb}}{m}\left(-\frac{m}{k}\right)e^{-\frac{k}{m}t} + C = -\frac{100 \text{ lb}}{k}e^{-\frac{k}{m}t} + C$

i.e., $e^{-\frac{k}{m}t}v = -\frac{100 \text{ lb}}{k}e^{-\frac{k}{m}t} + C$

Solve for v

$\Rightarrow v = -\frac{100 \text{ lb}}{k} + Ce^{\frac{k}{m}t}$

Next, we solve for the constant C, using our initial condition: "the boat starts from rest."

i.e. $v\left(0 \text{ sec}\right) = 0\frac{\text{ft}}{\text{sec}}$

$\Rightarrow 0\frac{\text{ft}}{\text{sec}} = v\left(0 \text{ sec}\right) = -\frac{100 \text{ lb}}{k} + Ce^{\frac{k}{m}\left(0 \text{ sec}\right)} = -\frac{100 \text{ lb}}{k} + C$

i.e., $0\frac{\text{ft}}{\text{sec}} = -\frac{100 \text{ lb}}{k} + C$

$\Rightarrow C = \frac{100 \text{ lb}}{k}$

$\Rightarrow v = -\frac{100 \text{ lb}}{k} + \frac{100 \text{ lb}}{k}e^{\frac{k}{m}t}$

Let's find the other constants

$\boxed{k}$

Recall: $R = kv$

$\Rightarrow k = \frac{R}{v}$

Recall: at $v = 40\ \frac{\text{ft}}{\text{sec}}$, $R = -80$ lb

$\Rightarrow k = \frac{-80 \text{ lb}}{40\ \frac{\text{ft}}{\text{sec}}} = -2\frac{\text{lb sec}}{\text{ft}}$

$\boxed{k = -2\frac{\text{lb sec}}{\text{ft}}}$

$\boxed{m}$

Recall: $w = mg$

$\Rightarrow m = \frac{w}{g} = \frac{-640 \text{ lb}}{-32\ \frac{\text{ft}}{\text{sec}^2}} = 20\frac{\text{lb sec}^2}{\text{ft}}$

$\boxed{m = 20\frac{\text{lb sec}^2}{\text{ft}}}$

$\Rightarrow v = -\frac{100 \text{ lb}}{\left(-2\frac{\text{lb sec}}{\text{ft}}\right)} + \frac{100 \text{ lb}}{\left(-2\frac{\text{lb sec}}{\text{ft}}\right)}e^{\frac{\left(-2\frac{\text{lb sec}}{\text{ft}}\right)}{\left(20\frac{\text{lb sec}^2}{\text{ft}}\right)}t} = 50\frac{\text{ft}}{\text{sec}} - 50\frac{\text{ft}}{\text{sec}}e^{-\frac{t}{10 \text{ sec}}}$

$\boxed{\text{i.e., } v\left(t\right) = 50\frac{\text{ft}}{\text{sec}} - 50\frac{\text{ft}}{\text{sec}}e^{-\frac{t}{10 \text{ sec}}}}$

To find the limiting velocity, we let $t \to \infty$

$\lim_{t\to\infty} v\left(t\right) = \lim_{t\to\infty}\left(50\frac{\text{ft}}{\text{sec}} - 50\frac{\text{ft}}{\text{sec}}e^{-\frac{t}{10 \text{ sec}}}\right) = 50\frac{\text{ft}}{\text{sec}}$

$\boxed{\text{i.e., } \lim_{t\to\infty} v\left(t\right) = 50\frac{\text{ft}}{\text{sec}}}$

2. Water at temperature $180\,^{\circ}\mathrm{F}$ cools in 20 minutes to $100\,^{\circ}\mathrm{F}$ in a room at temperature $40\,^{\circ}\mathrm{F}$.

   (a) Find the temperature T at any time $t \geq 0$ (Our answer will probably have to be expressed in terms of a natural logarithm)

   (b) Find the temperature at time $t = 40$ min

Let T be the temperature of the water at any time $t \geq 0$.

Let R be the room temperature

Let $t =$ time (in minutes)

By Newton's Law of Cooling (and Heating), the rate of change of water temperature with respect to time $\frac{dT}{dt}$, is proportional to the difference between the air and water temperatures $(T - R)$.

i.e., $\frac{dT}{dt} = k\,(T - R)$

$\Rightarrow \frac{dT}{dt} = kT - kR$

$\Rightarrow \frac{dT}{dt} - kT = -kR$

$\Rightarrow \frac{dT}{dt} + \underbrace{(-k)}_{P(t)}T = \underbrace{-kR}_{Q(t)}$

Our integrating factor is $e^{\int P(t)dt} = e^{\int (-k)dt} = e^{-kt}$

Multiplying both sides of our equation by the integrating factor, we have:

$e^{-kt}\frac{dT}{dt} - ke^{-kt}T = -kRe^{-kt}$

$\Rightarrow \frac{d}{dt}\left[e^{-kt}T\right] = -kRe^{-kt}$

$\Rightarrow \int \frac{d}{dt}\left[e^{-kt}T\right]dt = -kR\int e^{-kt}dt$

$\Rightarrow \left[e^{-kt}T\right] = -kR\left(-\frac{1}{k}e^{-kt}\right) = R\left(e^{-kt}\right) + C$

i.e., $e^{-kt}T = R\left(e^{-kt}\right) + C$

Solving for T, we have:

$T = R + Ce^{kt}$

Recall: $R = 40\,^{\circ}\mathrm{F}$

$\Rightarrow T = 40\,^{\circ}\mathrm{F} + Ce^{kt}$

Solve for C

Recall: When $t = 0$ minutes, $T = 180\,^\circ\mathrm{F}$

$\Rightarrow 180\,^\circ\mathrm{F} = T\,(0 \text{ min}) = 40\,^\circ\mathrm{F} + Ce^{k(0 \text{ min})} = 40\,^\circ\mathrm{F} + C$

i.e., $180\,^\circ\mathrm{F} = 40\,^\circ\mathrm{F} + C$

$\Rightarrow 140\,^\circ\mathrm{F} = C$

$\Rightarrow T = 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}e^{kt}$

Solve for k

Recall: When $t = 20$ min, $T = 100\,^\circ\mathrm{F}$

$\Rightarrow 100\,^\circ\mathrm{F} = T\,(20 \text{ min}) = 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}e^{k(20 \text{ min})}$

i.e., $100\,^\circ\mathrm{F} = 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}e^{k(20 \text{ min})}$

$\Rightarrow 60\,^\circ\mathrm{F} = 140\,^\circ\mathrm{F}e^{k(20 \text{ min})}$

$\Rightarrow \frac{60\,^\circ\mathrm{F}}{140\,^\circ\mathrm{F}} = e^{k(20 \text{ min})}$

$\Rightarrow \frac{3}{7} = e^{k(20 \text{ min})}$

$\Rightarrow \ln\left(\frac{3}{7}\right) = k\,(20 \text{ min})$

$\Rightarrow \frac{\ln\left(\frac{3}{7}\right)}{20 \text{ min}} = k$

$\Rightarrow T = 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}e^{\left(\frac{\ln\left(\frac{3}{7}\right)}{20 \text{ min}}\right)t}$

At $t = 40$ minutes,

$$T\,(40 \text{ minutes}) = 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}e^{\left(\frac{\ln\left(\frac{3}{7}\right)}{20 \text{ min}}\right)(40 \text{ minutes})} = 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}e^{\left(\ln\left(\frac{3}{7}\right)\right)(2)} = 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}e^{2\ln\left(\frac{3}{7}\right)}$$

$$= 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}e^{\ln\left(\frac{3}{7}\right)^2} = 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}\left(\frac{3}{7}\right)^2 = 40\,^\circ\mathrm{F} + 140\,^\circ\mathrm{F}\left(\frac{9}{49}\right)$$

$$= 40\,^\circ\mathrm{F} + 20\,^\circ\mathrm{F}\left(\frac{9}{7}\right) = 40\,^\circ\mathrm{F} + \frac{180}{7}\,^\circ\mathrm{F} = 40\,^\circ\mathrm{F} + \left(25 + \frac{5}{7}\right)\,^\circ\mathrm{F} = \left(65 + \frac{5}{7}\right)\,^\circ\mathrm{F} \approx 65.71\,^\circ\mathrm{F}$$

$T\,(40 \text{ minutes}) \approx 65.71\,^\circ\mathrm{F}$

3. The demand and supply of a certain commodity are given in terms of thousands of units, respectively, by:

$D = 60 - 9p(t) - 5p'(t)$; $\quad S = 180 - 3p(t) - 8p'(t)$

At time $t = 0$; the price of the commodity is 10 units

(a) Find the price at any later time and obtain its graph.

(b) Determine whether there is price stability and the equilibrium price if one exists.

Equating supply and demand, we have:

$3p'(t) - 6p(t) = 120$

$p'(t) - 2p(t) = 40$

$p'(t) + \underbrace{(-2)}_{P(t)} p(t) = \underbrace{40}_{Q(t)}$

Computing the integrating factor, we have: $e^{\int P(t)dt} = e^{\int (-2)dt} = e^{-2t}$

Multiplying both sides of the equation by the integrating factor, we have:

$e^{-2t}p'(t) - 2e^{-2t}p(t) = 40e^{-2t}$

$\Rightarrow \frac{d}{dt}\left[e^{-2t}p(t)\right] = 40e^{-2t}$

$\Rightarrow \int \frac{d}{dt}\left[e^{-2t}p(t)\right] dt = \int 40e^{-2t}dt$

$\Rightarrow e^{-2t}p(t) = 40\left(-\frac{1}{2}e^{-2t}\right) + C = -20e^{-2t} + C$

i.e., $e^{-2t}p(t) = -20e^{-2t} + C$

Solving for $p(t)$, we have:

$p(t) = -20 + Ce^{2t}$

Next, we find the value of the constant C

Recall: At time $t = 0$; the price of the commodity is 10 units

$\Rightarrow 10 = p(0) = -20 + Ce^{2(0)} = -20 + C$

i.e., $10 = -20 + C$

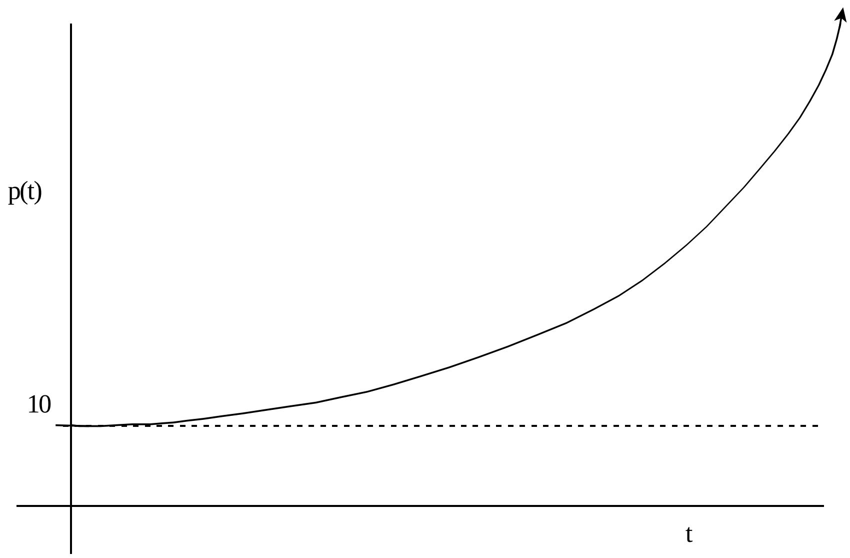
$\Rightarrow C = 30$

$$\boxed{p(t) = -20 + 30e^{2t}}$$

To determine whether there is price stability and an equilibrium price, we consider the limit of $p(t)$ as $t \to \infty$

$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \left(-20 + 30e^{2t}\right) = \infty$

i.e., $\lim_{t\to\infty} p(t) = \infty$



The market is **unstable.** There is no equilibrium price. The price will continue to increase without bound, until external factors bring the entire system to a halt.