MTH 3311 – Test #2 – Solutions

Spring 2018

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Directions: Do two of the three exercises.

1. A paratrooper and parachute weigh 160 lb. At the instant the parachute opens, she is traveling vertically downward at 40 $\frac{\text{ft}}{\text{sec}}$. If the air resistance varies directly as the instantaneous velocity, and the air resistance is 100 lb when the velocity is 20 $\frac{\text{ft}}{\text{sec}}$:

   a) Determine the velocity at any time t.

   b) Find the limiting velocity.

First, we establish our conventions regarding direction:

Positive Direction $\uparrow$

We will start our stopwatch at the instant that the parachute opens.

Thus, $v\left(0\text{ sec}\right) = -40\ \frac{\text{ft}}{\text{sec}}$ (The velocity is negative because the motion is in the downward direction.)

Let R be the resistance due to air.

When $v = -20\frac{\text{ft}}{\text{sec}}$, $R = 100$ lb (Because "For a velocity of $20\frac{\text{ft}}{\text{sec}}$ (in the *negative* direction), ... air resistance is 100 lb.")

Also: "The force of air resistance is proportional to the velocity" i.e. $R \propto v$

$\Rightarrow R = kv$, where k is the **constant of proportionality.**

For "future reference," we will find the constant of proportionality right now.

Recall: When $v = -20\frac{\text{ft}}{\text{sec}}$, $R = 100$ lb

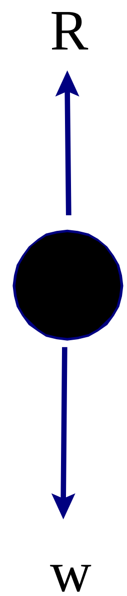
Also: $R = kv$

$\Rightarrow R = 100\text{ lb} = k\left(-20\frac{\text{ft}}{\text{sec}}\right)$

$\Rightarrow k = \frac{100\text{ lb}}{-20\frac{\text{ft}}{\text{sec}}} = -5\frac{\text{lb sec}}{\text{ft}}$

$\Rightarrow k = -5\frac{\text{lb sec}}{\text{ft}}$ (For "future reference")

Next: Since there is more than one force acting on the object, let's draw a force diagram of the object.



From the force diagram, the total force $F = w + R$,

where w is the **weight** of the object and R is the force on the object, due to air resistance.

Remark: To allow ourselves to model this relationship as a differential equation, we will employ a **standard trick:**

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Note well: When more than one force is acting on a free falling object, our approach will usually be to set F (the sum or all forces on the object) equal to ma.

$$\underbrace{\text{(sum of all forces)}}_{F} = \underbrace{ma}_{F}$$

This is a standard approach for velocity exercises!!!

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Our "Standard Trick" yields the equation $\underbrace{w + R}_{\text{Sum of all forces}} = \underbrace{ma}_{F}$ (Eq. 1)

Recall: acceleration is the derivative of velocity. i.e., $a = \frac{dv}{dt}$

Thus, Eq. 1 can be rendered:

$$\underbrace{w + kv}_{w+R} = \underbrace{m\frac{dv}{dt}}_{ma}$$

This is a differential equation in v.

Let's solve it!

$-m\frac{dv}{dt} + kv = -w$

$$\Rightarrow \frac{dv}{dt} + \underbrace{\left(-\frac{k}{m}\right)}_{P(t)} v = \underbrace{\frac{w}{m}}_{Q(t)}$$

Compute the integrating factor, $e^{\int P(t)dt} = e^{\int\left(-\frac{k}{m}\right)dt} = e^{-\frac{k}{m}t}$

Multiplying both sides by the integrating factor, we have:

$e^{-\frac{k}{m}t}\frac{dv}{dt} + \left(-\frac{k}{m}\right)e^{-\frac{k}{m}t}v = \frac{w}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right] = \frac{w}{m}e^{-\frac{k}{m}t}$

$\Rightarrow \int\left(\frac{d}{dt}\left[e^{-\frac{k}{m}t}v\right]\right)dt = \int \frac{w}{m}e^{-\frac{k}{m}t}dt$

$\Rightarrow e^{-\frac{k}{m}t}v = \frac{w}{m}\left(-\frac{m}{k}\right)e^{-\frac{k}{m}t} = -\frac{w}{k}e^{-\frac{k}{m}t} + C$

i.e. $e^{-\frac{k}{m}t}v = -\frac{w}{k}e^{-\frac{k}{m}t} + C$

$\Rightarrow v = -\frac{w}{k} + e^{\frac{k}{m}t}C$

Now, let's find the constant C

Recall: $v\,(0 \text{ sec}) = -40\frac{\text{ft}}{\text{sec}}$

$\Rightarrow -40\frac{\text{ft}}{\text{sec}} = v\,(0 \text{ sec}) = -\frac{w}{k} + e^{\frac{k}{m}(0 \text{ sec})}C = -\frac{w}{k} + C$

i.e. $-40\frac{\text{ft}}{\text{sec}} = -\frac{w}{k} + C$

$\Rightarrow C = \frac{w}{k} - 40\frac{\text{ft}}{\text{sec}}$

$\Rightarrow v = -\frac{w}{k} + \left(\frac{w}{k} - 40\frac{\text{ft}}{\text{sec}}\right)e^{\frac{k}{m}t}$

To find $\frac{w}{k}$, recall two things:

First, $k = -5\frac{\text{lb sec}}{\text{ft}}$

Next, the weight, $w = -160$ lb.

Thus, $\frac{w}{k} = \frac{-160 \text{ lb}}{-5\frac{\text{lb sec}}{\text{ft}}} = 32\ \frac{\text{ft}}{\text{sec}}$

i.e., $\frac{w}{k} = 32\ \frac{\text{ft}}{\text{sec}}$

Finally, we want to find $\frac{k}{m}$.

Note that $w = mg$, where g is the acceleration due to gravity.

$\Rightarrow m = \frac{w}{g} = \frac{-160 \text{ lb}}{-32\frac{\text{ft}}{\text{sec}^2}} = 5\frac{\text{lb sec}^2}{\text{ft}}$

i.e., $m = 5\frac{\text{lb sec}^2}{\text{ft}}$

Hence, observe that $\frac{k}{m} = \frac{-5\frac{\text{lb sec}}{\text{ft}}}{5\frac{\text{lb sec}^2}{\text{ft}}} = -\frac{1}{\text{sec}}$

Therefore, **velocity** is given by: $v\,(t) = -32\frac{\text{ft}}{\text{sec}} + \left(32\frac{\text{ft}}{\text{sec}} - 40\frac{\text{ft}}{\text{sec}}\right)e^{-\frac{1}{\text{sec}}t} = -32\frac{\text{ft}}{\text{sec}} - 8\frac{\text{ft}}{\text{sec}}e^{-\frac{1}{\text{sec}}t}$

a) Therefore, **velocity** is given by: $v\,(t) = -32\frac{\text{ft}}{\text{sec}} - 8\frac{\text{ft}}{\text{sec}}e^{-\frac{1}{\text{sec}}t}$

b) Find the limiting velocity

limiting velocity $= \lim_{t\to\infty} v\,(t) = \lim_{t\to\infty}\left(v\,(t) = -32\frac{\text{ft}}{\text{sec}} - 8\frac{\text{ft}}{\text{sec}}e^{-\frac{1}{\text{sec}}t}\right) = -32\frac{\text{ft}}{\text{sec}}$

The **limiting velocity** is given by: $v\,(t) = -32\frac{\text{ft}}{\text{sec}}$

2. Water at 35 °F "heats up" in 30 minutes to 55 °F in a room of temperature of 75 °F.

   a) Find the temperature of the water at time t.

   b) Find the water temperature at $t = 60\,\text{min}$

   c) What is the temperature as $t \to \infty$?

Let $T =$ temperature of the water

$t =$ time

$r =$ room temperature $= 75\,^\circ\text{F}$

According to **Newton's Law of Cooling,** the rate of change of water temperature, with respect to time, is proportional to the **difference** between the air and water temperatures.

Hence, $\frac{dT}{dt} = k\,(T - r)$ (Where k is the constant of proportionality)

$\Rightarrow \frac{dT}{(T-r)} = kdt$

$\Rightarrow \int \frac{1}{(T-r)} dT = \int kdt$

$\Rightarrow \ln|T - r| = kt + C$ (We can assume that $(T - r) < 0$, since $T < r$ initially. Hence, if we want to remove the absolute value bars, we have to acknowledge that, since $(T - r) < 0$, it follows that $|T - r| = -(T - r) = (r - T)$.)

Thus, we have: $\Rightarrow \ln|T - r| = kt + C$

$\Rightarrow \ln(r - T) = kt + C$

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Alternatively, since $T < r$ initially (i.e., $r > T$), we could take the equation, $\frac{dT}{dt} = k\,(T - r)$ and manipulate it as follows: $\frac{dT}{dt} = k\,(T - r) \Rightarrow \frac{dT}{dt} = -k\,(r - T)$, eliminating the need to worry about absolute value.

$\Rightarrow \frac{dT}{(r-T)} = -kdt$

$\Rightarrow \int \frac{1}{(r-T)} dT = \int -kdt$

$\Rightarrow -\ln|r - T| = -kt + C$

$\Rightarrow \ln|r - T| = kt + C$ (Since $(r - T) > 0$, it follows that $|r - T| = (r - T)$.)

Thus, we have: $\ln(r - T) = kt + C$

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$\Rightarrow e^{\ln(r-T)} = e^{kt+C} = C_1 e^{kt}$

i.e., $r - T = C_1 e^{kt}$

(Since $r = 75\,^\circ\text{F}$ is constant, we put the value in here.)

i.e., $75\,^\circ\text{F} - T = C_1 e^{kt}$

$\Rightarrow 75\,^\circ\text{F} - C_1 e^{kt} = T$

$\Rightarrow T = 75\,^\circ\text{F} - C_1 e^{kt}$

Notice that we have two constants to evaluate.

Therefore, we need two initial conditions

Recall: at $t = 0\,\text{min}$, $T = 35\,°\text{F}$

$\Rightarrow 35\,°\text{F} = T\,(0\,\text{min}) = 75\,°\text{F} - C_1 e^{k(0\,\text{min})}$

$\Rightarrow 35\,°\text{F} = 75\,°\text{F} - C_1$

$\Rightarrow -40\,°\text{F} = -C_1$

$\Rightarrow C_1 = 40\,°\text{F}$

Thus, $T\,(t) = 75\,°\text{F} - 40\,°\text{F}e^{kt}$

Recall Also: at $t = 30\,\text{min}$, $T = 55\,°\text{F}$

This yields: $55\,°\text{F} = T\,(30\,\text{min}) = 75\,°\text{F} - 40\,°\text{F}e^{k(30\,\text{min})}$

i.e., $55\,°\text{F} = 75\,°\text{F} - 40\,°\text{F}e^{k(30\,\text{min})}$

$\Rightarrow -20\,°\text{F} = -40\,°\text{F}e^{k(30\,\text{min})}$

$\Rightarrow \frac{-20\,°\text{F}}{-40\,°\text{F}} = e^{k(30\,\text{min})}$

$\Rightarrow \ln\left(\frac{1}{2}\right) = \ln\left(e^{k(30\,\text{min})}\right)$

$\Rightarrow \ln\left(\frac{1}{2}\right) = k\,(30\,\text{min})$

$\Rightarrow \frac{\ln\left(\frac{1}{2}\right)}{(30\,\text{min})} = k$

Since $\ln\left(\frac{1}{2}\right) = -\ln(2)$, we have:

$\Rightarrow k = -\frac{\ln(2)}{30\,\text{min}}$

$$\boxed{\Rightarrow T\,(t) = 75\,°\text{F} - 40\,°\text{F}e^{-\frac{\ln(2)}{30\,\text{min}}t}}$$

b) Find the water temperature at $t = 60\,\text{min}$

$$T\,(60\,\text{min}) = 75\,°\text{F} - 40\,°\text{F}e^{-\frac{\ln(2)}{30\,\text{min}}(60\,\text{min})} = 75\,°\text{F} - 40\,°\text{F}e^{-2\ln(2)} = 75\,°\text{F} - 40\,°\text{F}e^{\ln\left(2^{-2}\right)}$$

$$= 75\,°\text{F} - 40\,°\text{F}\left(2^{-2}\right) = 75\,°\text{F} - 10\,°\text{F} = 65\,°\text{F}$$

$$\boxed{T\,(60\,\text{min}) = 65\,°\text{F}}$$

c) What is the temperature as $t \to \infty$?

$$\lim_{t\to\infty} T\,(t) = \lim_{t\to\infty}\left(75\,°\text{F} - 40\,°\text{F}e^{-\frac{\ln(2)}{30\,\text{min}}t}\right) = 75\,°\text{F}$$

$$\boxed{\text{i.e., } \lim_{t\to\infty} T\,(t) = 75\,°\text{F}}$$

3. The demand and supply of a certain commodity are given in terms of thousands of units, respectively, by:

$$D = 50 + 12p(t) + 2p'(t); \quad S = 450 - 8p(t) - 2p'(t).$$

At $t = 0$, the price of the commodity is 40 units.

a) Find the price at any later time and obtain its graph.

b) determine whether there is price stability and the equilibrium price if one exists.

Equating supply and demand, we have:

$50 + 12p(t) + 2p'(t) = 450 - 8p(t) - 2p'(t)$

$\Rightarrow 4p'(t) + 20p(t) = 400$

$\Rightarrow p'(t) + \underbrace{5}_{P(t)} p(t) = \underbrace{100}_{Q(t)}$

Our integrating factor is $e^{\int P(t)dt} = e^{\int 5dt} = e^{5t}$

Multiplying both sides by the integrating factor, e^{5t}, we have:

$e^{5t}p'(t) + 5e^{5t}p(t) = 100e^{5t}$

$\Rightarrow \frac{d}{dt}\left[e^{5t}p(t)\right] = 100e^{5t}$ Integrating, we have:

$\Rightarrow \int \left(\frac{d}{dt}\left[e^{5t}p(t)\right]\right) dt = \int 100e^{5t}dt$

$\Rightarrow e^{5t}p(t) = 100\left(\frac{1}{5}\right)e^{5t} + C = 20e^{5t} + C$

i.e., $e^{5t}p(t) = 20e^{5t} + C$

$\Rightarrow p(t) = 20 + e^{-5t}C$

To find the constant C, we use our initial condition $p(0) = 40$ (Because "At $t = 0$, the price of the commodity is 40 units.")

$\Rightarrow 40 = p(0) = 20 + e^{-5(0)}C = 20 + C$

i.e., $40 = 20 + C$

i.e., $20 = C$

Hence, $p(t) = 20 + 20e^{-5t}$ is the price at any time t.

To graph the function, let's consider the derivative.

$p'(t) = -100e^{-5t}$

Note that $p'(t) < 0$ for all values of t, since $e^{\text{ham sandwich}}$ is always positive.

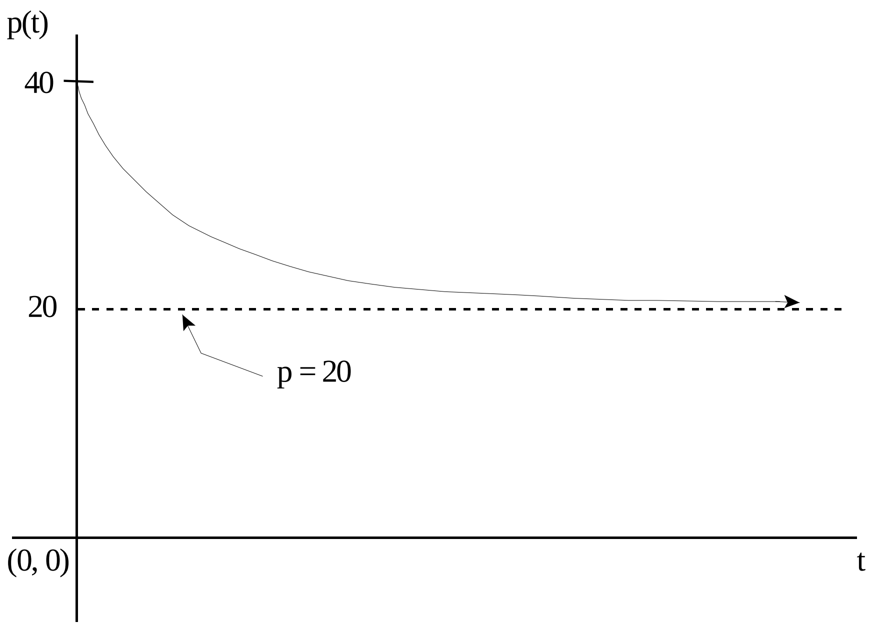
Hence, the graph of $p(t)$ is decreasing.

Next, let's consider the graph of $p(t)$ as $t \to \infty$.

$\lim_{t\to\infty} p(t) = \lim_{t\to\infty}\left(20 + 20e^{-5t}\right) = 20 + 0 = 20$

i.e., $\lim_{t\to\infty} p(t) = 20$

The graph of $y = p(t)$ is given below:



The market is **stable.** The equilibrium price is 20 units. The price will continue to decrease toward the equilibrium price $p_e = 20$.