MTH 3331 - Practice Test #3a - Solutions

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1. Solve, first using Undetermined Coefficients, then using Variation of Parameters:

   $x^2y'' + 3xy' + y = 2x$

   This is a nonhomogeous Euler's Equation.

   First, we find the solution to the corresponding complementary equation:

   $x^2y'' + 3xy' + y = 0.$

   Recall, that solutions of this type of equation are of the form: $y = x^\lambda$

   $$\Rightarrow \begin{array}{rcl} y & = & x^\lambda \\ y' & = & \lambda x^{\lambda-1} \\ y'' & = & \lambda(\lambda-1)x^{\lambda-2} \end{array}$$

   Substituting this into the equation $x^2y'' + 3xy' + y = 0$, we have:

   $x^2\left(\lambda(\lambda-1)x^{\lambda-2}\right) + 3x\left(\lambda x^{\lambda-1}\right) + x^\lambda = 0 \Rightarrow \lambda(\lambda-1)x^\lambda + 3\lambda x^\lambda + x^\lambda = 0$

   $\Rightarrow \lambda^2x^\lambda + 2\lambda x^\lambda + x^\lambda = 0$

   Since $x^\lambda \neq 0$, we have $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda = -1$; double root.

   Hence, our complementary solution is of the form: $y_c = c_1x^{-1} + c_2\ln(x)\,x^{-1}$

   To find the general solution, we can either us the Method of Undetermined Coefficients or the Method of Variation of Parameters. (on succeeding pages)

Using Method of Undetermined Coefficients

Since the original equation is of the form: $x^2y'' + 3xy' + y = 2x$,

We guess that $y_p = Ax$

$\Rightarrow y_p' = A$

$\Rightarrow y_p'' = 0$

Plugging these into the equation $x^2y'' + 3xy' + y = 2x$, we have:

$x^2y'' + 3xy' + y = x^2A \cdot 0 + 3xA + Ax = 2x$

$\Rightarrow (3A + A)\,x = 2x$

$\Rightarrow 4A = 2 \Rightarrow A = \frac{1}{2}$

$\Rightarrow y_p = Ax = \frac{1}{2}x$

Our general solution is: $y = y_p + y_c = \frac{1}{2}x + c_1x^{-1} + c_2\ln(x)\,x^{-1}$

Using Method of Variation of Parameters

To find the general solution, let $y = c_1(x)\,x^{-1} + c_2(x)\ln(x)\,x^{-1}$

Restriction #1: $c_1(x)$ and $c_2(x)$ are such that $y = c_1(x)\,x^{-1} + c_2(x)\ln(x)\,x^{-1}$ is a solution to the original, nonhomogeneous equation, $x^2y'' + 3xy' + y = 2x$

$\Rightarrow y' = c_1'(x)\,x^{-1} - c_1(x)\,x^{-2} + c_2'(x)\ln(x)\,x^{-1} + c_2(x)\frac{1}{x^2}(1 - \ln(x))$

Restriction #2, We'll let $c_1'(x)\,x^{-1} + c_2'(x)\ln(x)\,x^{-1} = 0$ (Note that this is equivalent to saying that $c_1'(x) + c_2'(x)\ln(x) = 0$, since $x^{-1} \neq 0$)

Thus we have: $y' = -c_1(x)\,x^{-2} + c_2(x)\frac{1}{x^2}(1 - \ln(x))$

$\Rightarrow y'' = -c_1'(x)\,x^{-2} + 2c_1(x)\,x^{-3} + c_2'(x)\frac{1}{x^2}(1 - \ln(x)) + c_2(x)\,x^{-3}[2\ln(x) - 3]$

Plugging into the equation $x^2y'' + 3xy' + y = 2x$, we have:

$$x^2\left(-c_1'(x)\,x^{-2} + 2c_1(x)\,x^{-3} + c_2'(x)\frac{1}{x^2}(1 - \ln(x)) + c_2(x)\,x^{-3}[2\ln(x) - 3]\right)$$
$$+3x\left(-c_1(x)\,x^{-2} + c_2(x)\frac{1}{x^2}(1 - \ln(x))\right) + c_1(x)\,x^{-1} + c_2(x)\ln(x)\,x^{-1} = 2x$$

$$\Rightarrow \left(-c_1'(x) + 2c_1(x)\,x^{-1} + c_2'(x)(1 - \ln(x)) + c_2(x)\,x^{-1}[2\ln(x) - 3]\right)$$
$$+3\left(-c_1(x)\,x^{-1} + c_2(x)\,x^{-1}(1 - \ln(x))\right) + c_1(x)\,x^{-1} + c_2(x)\ln(x)\,x^{-1} = 2x$$

By our second restriction, this becomes:

$$\left(2c_1(x)\,x^{-1} + c_2'(x) + c_2(x)\,x^{-1}[2\ln(x) - 3]\right) + 3\left(-c_1(x)\,x^{-1} + c_2(x)\,x^{-1}(1 - \ln(x))\right)$$
$$+c_1(x)\,x^{-1} + c_2(x)\ln(x)\,x^{-1} = 2x$$

Simplifying, we have: $c_2'(x) = 2x \Rightarrow c_2(x) = x^2 + C_3$

To find $c_1(x)$, we'll use Restriction #2, $c_1'(x) + c_2'(x)\ln(x) = 0$

Thus, we have: $c_1'(x) + 2x\ln(x) = 0 \Rightarrow c_1'(x) = -2x\ln(x)$

$\Rightarrow c_1(x) = -2\int x\ln(x)\,dx = -x^2\ln x + \frac{1}{2}x^2 + C_4 = x^2\left(\frac{1}{2} - \ln(x)\right) + C_4$

So we have: $c_1(x) = x^2\left(\frac{1}{2} - \ln(x)\right) + C_4$ and $c_2(x) = x^2 + C_3$

The solution to the original, nonhomogeneous equation, $x^2y'' + 3xy' + y = 2x$ is

$y = c_1(x)\,x^{-1} + c_2(x)\ln(x)\,x^{-1} = \left[x^2\left(\frac{1}{2} - \ln(x)\right) + C_4\right]x^{-1} + \left[x^2 + C_3\right]\ln(x)\,x^{-1}$

$= \frac{1}{2}x + C_4x^{-1} + C_3x^{-1}\ln(x)$

i.e., $y = \frac{1}{2}x + C_4x^{-1} + C_3x^{-1}\ln(x)$ is our solution.

2. Find the general solution of the equation: $y'' - 2y' + y = \frac{1}{x}e^x;\ x > 0$

First, we consider the corresponding complementary equation, $y'' - 2y' + y = 0$

This yields the auxilliary equation $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$ is a double root.

The complementary solution will be of the form $y = c_1 e^x + c_2 x e^x$

To find the solution to the original, nonhomogeoeous equation, $y'' - 2y' + y = \frac{1}{x}e^x$, we suppose that

$y = c_1(x) e^x + c_2(x) x e^x$ is the solution.

We have two arbitrary functions of x, $c_1(x)$, and $c_2(x)$. This means that we are allowed to make two restrictions on the functions.

Restriction #1 $c_1(x)$, and $c_2(x)$ are such that $y = c_1(x) e^x + c_2(x) x e^x$ is the solution to the original, nonhomogeoeous equation, $y'' - 2y' + y = \frac{1}{x}e^x$.

$\Rightarrow y' = c_1'(x) e^x + c_1(x) e^x + c_2'(x) x e^x + c_2(x)(1+x) e^x$

Restriction #2 We'll assume that $c_1'(x) e^x + c_2'(x) x e^x = 0$ (Note that this is equivalent to saying that $c_1'(x) + c_2'(x) x = 0$, since $e^x \neq 0$)

Thus we have: $y' = c_1(x) e^x + c_2(x)(1+x) e^x$

$\Rightarrow y'' = c_1'(x) e^x + c_1(x) e^x + c_2'(x)(1+x) e^x + c_2(x)(2+x) e^x$

To simplify this, we can rewrite y'' as $y'' = c_1'(x) e^x + c_1(x) e^x + c_2'(x) x e^x + c_2'(x) e^x + c_2(x)(2+x) e^x$ and apply Restriction #2.

$\Rightarrow y'' = c_1(x) e^x + c_2'(x) e^x + c_2(x)(2+x) e^x$

Plugging these into our original, nonhomogeoeous equation, $y'' - 2y' + y = \frac{1}{x}e^x$, we have:

$c_1(x) e^x + c_2'(x) e^x + c_2(x)(2+x) e^x - 2\left(c_1(x) e^x + c_2(x)(1+x) e^x\right) + c_1(x) e^x + c_2(x) x e^x = \frac{1}{x}e^x$

Simplifying, this becomes: $c_2'(x) e^x = \frac{1}{x}e^x$

$\Rightarrow c_2'(x) = \frac{1}{x} \Rightarrow c_2(x) = \ln(x) + C_3$

To find $c_1(x)$, we'll use Restriction #2, $c_1'(x) e^x + c_2'(x) x e^x = 0$

$c_1'(x) e^x + \left(\frac{1}{x}\right) x e^x = 0 \Rightarrow c_1'(x) e^x + e^x = 0 \Rightarrow c_1'(x) + 1 = 0$

$\Rightarrow c_1'(x) = -1 \Rightarrow c_1(x) = -x + C_4$

Thus $c_1(x) = -x + C_4$; and $c_2(x) = \ln(x) + C_3$

This yields the solution

$y = c_1(x)\, e^x + c_2(x)\, xe^x$

$\Rightarrow y = (-x + C_4)\, e^x + (\ln(x) + C_3)\, xe^x$

i.e. $y = -xe^x + x\ln(x)\, e^x + C_4 e^x + C_3 x e^e$ is the solution of the original, nonhomogeneous equation, $y'' - 2y' + y = \frac{1}{x}e^x$

3. Find the general solution of the equation $y'' + 8y' + 17y = x^2 + 3x + 2$

First, we consider the corresponding complementary equation, $y'' + 8y' + 17y = 0$

This yields the auxilliary equation $m^2 + 8m + 17 = 0 \Rightarrow m = \frac{-8 \pm \sqrt{(-8)^2 - 4(1)(17)}}{2(1)}$

$\Rightarrow m = -4 \pm i$

Thus the complementary solution is of the form: $y_c = c_1 e^{(-4+i)x} + c_2 e^{(-4-i)x} \Rightarrow y_c = e^{-4x}\left(c_1 e^{ix} + c_2 e^{-ix}\right)$

$\Rightarrow y_c = e^{-4x}\left(A\cos(x) + B\sin(x)\right)$

To find the solution to the original, nonhomogeoeous equation:

$y'' + 8y' + 17y = x^2 + 3x + 2,$

we use the Method of Undetermined Coefficients. (This is our best bet whenever the right hand side is a polynomial.)

Since the highest degree of the right hand side is 2 and the equation contains the function y as well as its derivatives, we assume that our particular solution y_p has the form

$$\begin{aligned} y_p &= Ax^2 + Bx + C \\ \Rightarrow\ y_p' &= 2Ax + B \\ \Rightarrow\ y_p'' &= 2A \end{aligned}$$

Next, we plug these into the origianal equation and solve for the constants.

$y_p'' + 8y_p' + 17y_p = x^2 + 3x + 2$

$\Rightarrow (2A) + 8(2Ax + B) + 17(Ax^2 + Bx + C) = x^2 + 3x + 2$

$\Rightarrow 17Ax^2 + (16A + 17B)x + (2A + 8B + 17C) = x^2 + 3x + 2$

Comparing the coefficeints of the different powers of x, we have:

$17A = 1 \Rightarrow A = \frac{1}{17}$

$16A + 17B = 3 \Rightarrow 16\left(\frac{1}{17}\right) + 17B = 3 \Rightarrow 17B = 3 \Rightarrow B = \frac{35}{289}$

$2A + 8B + 17C = 2 \Rightarrow 2\left(\frac{1}{17}\right) + 8\left(\frac{35}{289}\right) + 17C = 2$

$\Rightarrow 17C = \frac{264}{289} \Rightarrow C = \frac{264}{4913}$

Thus, $y_p = Ax^2 + Bx + C = \frac{1}{17}x^2 + \frac{35}{289}x + \frac{264}{4913}$

Thus, our solution is:

$$y = y_c + y_p = e^{-4x}\left(A\cos(x) + B\sin(x)\right) + \frac{1}{17}x^2 + \frac{35}{289}x + \frac{264}{4913}$$