MTH 3331 - Test #3 - Solutions

Spring 2001

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1. Solve, using the Method of Undetermined Coefficients: $y'' - 4y' + 4y = e^{2x}$

   First, find the solution to the complementary equation $y'' - 4y' + 4y = 0$

   $\Rightarrow m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2$ is a double root.

   $\Rightarrow y_c = c_1 e^{2x} + c_2 x e^{2x}$

   For the particular solution, we imagine that $y_p = Ax^2 e^{2x}$

   (We use this instead of $y_p = Ae^{2x}$, because because e^{2x} and xe^{2x} are already part of the **homogeneous** solution.)

   $\Rightarrow y_p' = 2Axe^{2x} + 2Ax^2 e^{2x}$

   $\Rightarrow y_p'' = 2Ae^{2x} + 4Axe^{2x} + 4Axe^{2x} + 4Ax^2 e^{2x}$

   Simplifying, we have: $y_p'' = 2Ae^{2x} + 8Axe^{2x} + 4Ax^2 e^{2x}$

   To find A, we plug into the original equation, $y'' - 4y' + 4y = e^{2x}$.

   $$\Rightarrow \underbrace{2Ae^{2x} + 8Axe^{2x} + 4Ax^2 e^{2x}}_{y''} \underbrace{- 4\left(2Axe^{2x} + 2Ax^2 e^{2x}\right)}_{-4y'} + \underbrace{4\left(Ax^2 e^{2x}\right)}_{4y} = e^{2x}$$

   $\Rightarrow 2Ae^{2x} = e^{2x} \Rightarrow A = \frac{1}{2} \Rightarrow y_p = \frac{1}{2}x^2 e^{2x}$

   The solution to the original equation is: $y = y_p + y_c$

   $$\boxed{\Rightarrow y = \frac{1}{2}x^2 e^{2x} + c_1 e^{2x} + c_2 x e^{2x}}$$

2. Solve, using Variation of Parameters: $y'' + y = \csc(x)$

   First, find the solution to the complementary equation $y'' + y = 0$

   $\Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i$

   $\Rightarrow y_c = c_1 e^{ix} + c_2 e^{-ix} = A\cos(x) + B\sin(x)$

   To find the general solution, we let $y = A(x)\cos(x) + B(x)\sin(x)$

RESTRICTION #1 $A(x)$ and $B(x)$ are such that $y = A(x)\cos(x) + B(x)\sin(x)$ actually IS a solution to the original equation $y'' + y = e^{2x}$

$\Rightarrow y' = A'(x)\cos(x) - A(x)\sin(x) + B'(x)\sin(x) + B(x)\cos(x)$

RESTRICTION #2 $A'(x)\cos(x) + B'(x)\sin(x) = 0$

Simplifying, y', we have: $y' = -A(x)\sin(x) + B(x)\cos(x)$

$\Rightarrow y'' = -A'(x)\sin(x) - A(x)\cos(x) + B'(x)\cos(x) - B(x)\sin(x)$

Plugging into the original equation, $y'' + y = \csc(x)$, we have:

$$\begin{array}{lclclclcl} y'' & = & -A'(x)\sin(x) & - & A(x)\cos(x) & + & B'(x)\cos(x) & - & B(x)\sin(x) & & \\ y & = & & & A(x)\cos(x) & & & + & B(x)\sin(x) & & \\ \hline\hline y''+y & = & -A'(x)\sin(x) & & & + & B'(x)\cos(x) & & & = & \csc(x) \end{array}$$

Using this in conjunction with restriction #2, we have:

$$\begin{array}{rlclcl} & -A'(x)\sin(x) & + & B'(x)\cos(x) & = & \csc(x) \\ \tan(x) & [A'(x)\cos(x) & + & B'(x)\sin(x)\} & = & \tan(x)[0] \\ \hline\hline & & & B'(x)\left[\frac{\sin^2(x)}{\cos(x)} + \cos(x)\right] & = & \csc(x) \end{array}$$

$\Rightarrow B'(x)\left[\frac{\sin^2(x)+\cos^2(x)}{\cos(x)}\right] = \csc(x) \Rightarrow B'(x)\left[\frac{1}{\cos(x)}\right] = \csc(x) \Rightarrow B'(x) = \cot(x)$

$\Rightarrow B(x) = \ln|\sin(x)| + C_3$

Recall:

$$\begin{array}{lcrlclcl} y''+y & = & & -A'(x)\sin(x) & + & B'(x)\cos(x) & = & \csc(x) \\ & & -\cot(x) & [A'(x)\cos(x) & + & B'(x)\sin(x) & = & 0 \\ \hline\hline & & & -A'(x)\left[\frac{\cos^2(x)}{\sin(x)} + \sin(x)\right] & & & = & \csc(x) \end{array}$$

$\Rightarrow -A'(x)\left[\frac{\cos^2(x)+\sin^2(x)}{\sin(x)}\right] = \csc(x) \Rightarrow -A'(x)\left[\frac{1}{\sin(x)}\right] = \csc(x) \Rightarrow -A'(x) = 1$

$\Rightarrow A'(x) = -1$

$\Rightarrow A(x) = -x + C_4$

The solution to the original equation, $y'' + y = \csc(x)$ is

$y = A(x)\cos(x) + B(x)\sin(x) \Rightarrow y = (-x + C_4)\cos(x) + (\ln|\sin(x)| + C_3)\sin(x)$

$$\boxed{y = -x\cos(x) + \ln|\sin(x)|\sin(x) + C_4\cos(x) + C_3\sin(x)}$$

3. Solve, first using Undetermined Coefficients, then using Variation of Parameters:

$x^2y'' + 4xy' - 4y = 2x$

First, find the solution to the corresponding homogeneous equation, $x^2y''+4xy'-4y = 0$

This is an Euler's Equation, so we expect the homogeneous soltuion to be of the form, $y = x^{\lambda}$

$\Rightarrow y' = \lambda x^{\lambda-1}$

$\Rightarrow y'' = \lambda(\lambda-1)x^{\lambda-2} = \left(\lambda^2-\lambda\right)x^{\lambda-2}$

Plugging into the equation $x^2y'' + 4xy' - 4y = 0$, we have:

$$\underbrace{x^2\left(\lambda^2-\lambda\right)x^{\lambda-2}}_{x^2y''} + \underbrace{4x\left(\lambda x^{\lambda-1}\right)}_{4xy'} - \underbrace{4x^{\lambda}}_{4y} = 0$$

$\Rightarrow \left(\lambda^2-\lambda\right)x^{\lambda} + 4\lambda x^{\lambda} - 4x^{\lambda} = 0 \Rightarrow \lambda^2x^{\lambda} + 3\lambda x^{\lambda} - 4x^{\lambda} = 0$

$\Rightarrow \lambda^2 + 3\lambda - 4 = 0 \Rightarrow (\lambda+4)(\lambda-1) \Rightarrow \lambda_1 = -4; \lambda_2 = 1$

$\Rightarrow y_c = c_1x^{-4} + c_2x$

To find the general solution, we can either us the Method of Undetermined Coefficients or the Method of Variation of Parameters. (on succeeding pages)

Using Method of Undetermined Coefficients

Since the original equation is of the form: $x^2y'' + 4xy' - 4y = 2x$,

We guess that $y_p = Ax$

However, this is essentially the same as one of the independent solutions of the complementary equation c_2x.

So we modify our guess: $y_p = A\ln(x)\,x$

$\Rightarrow y_p' = A\left(\ln(x)\cdot 1 + x\cdot\frac{1}{x}\right) = A(\ln(x)+1)$

$\Rightarrow y_p'' = A\frac{1}{x}$

Plugging these into the equation $x^2y'' + 4xy' - 4y = 2x$, we have:

$x^2y'' + 4xy' - 4y = x^2A\frac{1}{x} + 4xA(\ln(x)+1) - 4A\ln(x)\,x$

$= (A+4A)\,x + (4A-4A)\ln(x)\,x = 5Ax = 2x$

$\Rightarrow 5A = 2 \Rightarrow A = \frac{2}{5}$

$\Rightarrow y_p = A\ln(x)\,x = \frac{2}{5}\ln(x)\,x$

Our general solution is: $y = y_p + y_c = \frac{2}{5}\ln(x)\,x + c_1x^{-4} + c_2x$

Using Method of Variation of Parameters

To find the general solution, let $y = c_1(x)\,x^{-4} + c_2(x)\,x$

RESTRICTION #1: $c_1(x) + c_2(x)$ are such that $y = c_1(x)\,x^{-4} + c_2(x)\,x$ actually IS a solution to the original equation $x^2y'' + 4xy' - 4y = 2x$.

$\Rightarrow y' = c_1'(x)\,x^{-4} - 4c_1(x)\,x^{-5} + c_2'(x)\,x + c_2(x)$

RESTRICTION #2: $c_1'(x)\,x^{-4} + c_2'(x)\,x = 0$

$\Rightarrow y' = -4c_1(x)\,x^{-5} + c_2(x)$

$\Rightarrow y'' = -4c_1'(x)\,x^{-5} + 20c_1(x)\,x^{-6} + c_2'(x)$

Plug these into the original equation, $x^2y'' + 4xy' - 4y = 2x$.

$$\begin{array}{lcccccccccc}
x^2y'' & = & -4c_1'(x)\,x^{-3} & + & 20c_1(x)\,x^{-4} & + & c_2'(x)\,x^2 & & & & \\
+4xy' & = & & - & 16c_1(x)\,x^{-4} & & & + & 4c_2(x)\,x & & \\
-4y & = & & - & 4c_1(x)\,x^{-4} & & & - & 4c_2(x)\,x & & \\
\hline\hline
x^2y'' + 4xy' - 4y & = & -4c_1'(x)\,x^{-3} & & & + & c_2'(x)\,x^2 & & & = & 2x
\end{array}$$

Combining this last equation with our second restriction, we have:

$$\begin{array}{cccccc}
 & -4c_1'(x)\,x^{-3} & + & c_2'(x)\,x^2 & = & 2x \\
-x & [c_1'(x)\,x^{-4} & + & c_2'(x)\,x] & = & -x\,[0] \\
\hline\hline
 & -5c_1'(x)\,x^{-3} & & & = & 2x
\end{array}$$

i.e., $-5c_1'(x)\,x^{-3} = 2x \Rightarrow c_1'(x) = -\frac{2}{5}x^4 \Rightarrow c_1(x) = -\frac{2}{25}x^5 + C_3$

Recall:

$$\begin{array}{lccccccc}
x^2y'' + 4xy' - 4y & = & & -4c_1'(x)\,x^{-3} & + & c_2'(x)\,x^2 & = & 2x \\
\text{second restriction} & & 4x & [c_1'(x)\,x^{-4} & + & c_2'(x)\,x] & = & 4x\,[0] \\
\hline\hline
 & & & & & 5c_2'(x)\,x^2 & = & 2x
\end{array}$$

i.e., $5c_2'(x)\,x^2 = 2x \Rightarrow c_2'(x) = \frac{2}{5}x^{-1} \Rightarrow c_2(x) = \frac{2}{5}\ln|x| + C_4$

So the solution to the original equation, $x^2y'' + 4xy' - 4y = 2x$, is

$y = c_1(x)\,x^{-4} + c_2(x)\,x \Rightarrow y = \left(-\frac{2}{25}x^5 + C_3\right)x^{-4} + \left(\frac{2}{5}\ln|x| + C_4\right)x$

$= \frac{2}{5}x\ln|x| + C_3x^{-4} + \left(-\frac{2}{25} + C_4\right)x = \frac{2}{5}x\ln|x| + C_3x^{-4} + C_5x$

$$\boxed{\Rightarrow y = \frac{2}{5}x\ln|x| + C_3x^{-4} + C_5x}$$