MTH 3311 Practice Test #4

SPRING 2001

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Practice Test #4 - Solutions

1. $y'' + 4y = 4\cos(2t)$; $y(0) = 0$; $y'(0) = 6$

(a) 1. Take the Laplace Transform of each side.

$\Rightarrow \pounds[y'' + 4y] = \pounds[4\cos(2t)]$

$\Rightarrow \pounds[y''] + 4\pounds[y] = 4\pounds[\cos(2t)]$

$$\Rightarrow \underbrace{s^2\mathbf{Y}(s) - sy(0) - y'(0)}_{\pounds[y''] \text{ (formula 20)}} + \underbrace{4\mathbf{Y}(s)}_{\pounds[y]} = 4 \cdot \underbrace{\frac{s}{s^2+4}}_{\frac{\pounds[\cos(2t)]}{\text{(formula 9)}}}$$

$\Rightarrow s^2\mathbf{Y}(s) - s\cdot(0) - 6 + 4\mathbf{Y}(s) = 4\frac{s}{s^2+4}$ (By plugging in our initial conditions)

$\Rightarrow s^2\mathbf{Y}(s) - 6 + 4\mathbf{Y}(s) = 4\frac{s}{s^2+4}$

2. Solve for $\mathbf{Y}(s)$

$s^2\mathbf{Y}(s) + 4\mathbf{Y}(s) = 4\frac{s}{s^2+4} + 6 \Rightarrow (s^2+4)\mathbf{Y}(s) = \frac{4s}{s^2+4} + 6$

$\Rightarrow \mathbf{Y}(s) = \frac{4s}{(s^2+4)^2} + \frac{6}{s^2+4}$

3. Solve for y, by computing the Laplace transform inverse of each side.

$\Rightarrow \pounds^{-1}[\mathbf{Y}(s)] = \pounds^{-1}\left[\frac{4s}{(s^2+4)^2} + \frac{6}{s^2+4}\right]$

$\Rightarrow y = \pounds^{-1}[\mathbf{Y}(s)] = \pounds^{-1}\left[\frac{4s}{(s^2+4)^2} + \frac{6}{s^2+4}\right] = \pounds^{-1}\left[\frac{4s}{(s^2+4)^2}\right] + \pounds^{-1}\left[\frac{6}{s^2+4}\right]$

$$= \pounds^{-1}\left[\frac{(2)(2)s}{(s^2+2^2)^2}\right] + 3\pounds^{-1}\left[\frac{2}{(s^2+2^2)}\right] = \underbrace{t\sin(2t)}_{\text{formula 13}} + 3\underbrace{\sin(2t)}_{\text{formula 6}}$$

i.e., $y = t\sin(2t) + 3\sin(2t)$

2. $y'' + 3y' + 2y = 0;\quad y(0) = 1;\quad y'(0) = 1$

(a) 1. Take the Laplace Transform of each side.

$$\Rightarrow \pounds\left[y'' + 3y' + 2y\right] = \pounds\left[0\right]$$

$$\Rightarrow \pounds\left[y''\right] + 3\pounds\left[y'\right] + 2\pounds\left[y\right] = \pounds\left[0\right]$$

$$\Rightarrow \underbrace{s^2\mathbf{Y}(s) - sy(0) - y'(0)}_{\pounds[y'']\ \text{(formula 20)}} + \ 3\underbrace{s\mathbf{Y}(s) - y(0)}_{\pounds[y']\ \text{formula 19}} + \ 2\underbrace{\mathbf{Y}(s)}_{\pounds[y]} = 0$$

2. Solve for $\mathbf{Y}(s)$

$$\Rightarrow \left(s^2 + 3s + 2\right)\mathbf{Y}(s) - (s+3)\,y(0) - y'(0) = 0$$

$$\Rightarrow \left(s^2 + 3s + 2\right)\mathbf{Y}(s) - (s+3) - 1 = 0 \ \ \text{(plugging in initial values)}$$

$$\Rightarrow \left(s^2 + 3s + 2\right)\mathbf{Y}(s) - (s+4) = 0 \Rightarrow \left(s^2 + 3s + 2\right)\mathbf{Y}(s) = (s+4)$$

$$\Rightarrow \mathbf{Y}(s) = \tfrac{(s+4)}{(s^2+3s+2)}$$

3. Solve for y, by computing the Laplace transform inverse of each side.

$$\Rightarrow y = \pounds^{-1}\left[\mathbf{Y}(s)\right] = \pounds^{-1}\left[\tfrac{(s+4)}{(s^2+3s+2)}\right]$$

$$= \underbrace{\pounds^{-1}\left[\frac{(s+4)}{(s+2)(s+1)}\right] = \pounds^{-1}\left[-\frac{2}{s+2} + \frac{3}{s+1}\right]}_{\text{By Partial Fraction Decomposition}}$$

$$= -2\pounds^{-1}\left[\tfrac{1}{s+2}\right] + 3\pounds^{-1}\left[\tfrac{1}{s+1}\right]$$

$$= -2\pounds^{-1}\left[\tfrac{1}{s-(-2)}\right] + 3\pounds^{-1}\left[\tfrac{1}{s-(-1)}\right] = -2\ \underbrace{e^{-2t}}_{\text{formula 4}} \ + \ 3\ \underbrace{e^{-t}}_{\text{formula 4}}$$

i.e., $y = -2e^{-2t} + 3e^{-t}$

3. $y''' - 27y = 0;\;\; y(0) = -1;\;\; y'(0) = 6;\;\; y''(0) = 18$

(a) 1. Take the Laplace Transform of each side.

$$\Rightarrow \pounds\left[y''' - 27y\right] = \pounds\left[0\right]$$

$$\Rightarrow \pounds\left[y'''\right] - 27\pounds\left[y\right] = \pounds\left[0\right]$$

$$\Rightarrow \underbrace{s^3\mathbf{Y}(s) - s^2y(0) - sy'(0) - y''(0)}_{\pounds[y'''] \text{ (formula 21)}} - 27\underbrace{\mathbf{Y}(s)}_{\pounds[y]} = 0$$

2. Solve for $\mathbf{Y}(s)$

$$\Rightarrow (s^3 - 27)\,\mathbf{Y}(s) - s^2y(0) - sy'(0) - y''(0) = 0$$

$$\Rightarrow (s^3 - 27)\,\mathbf{Y}(s) - s^2(-1) - s(6) - 18 = 0$$

$$\Rightarrow (s^3 - 27)\,\mathbf{Y}(s) + s^2 - 6s - 18 = 0$$

$$\Rightarrow (s^3 - 27)\,\mathbf{Y}(s) = -s^2 + 6s + 18$$

$$\Rightarrow \mathbf{Y}(s) = \frac{-s^2+6s+18}{s^3-27} \Rightarrow \mathbf{Y}(s) = -\frac{s^2-6s-18}{s^3-27}$$

3. Solve for y, by computing the Laplace transform inverse of each side.

$$\Rightarrow y = \pounds^{-1}\left[\mathbf{Y}(s)\right] = \pounds^{-1}\left[\frac{-s^2+6s+18}{(s-3)(s^2+3s+9)}\right] = \pounds^{-1}\left[-\frac{s^2-6s-18}{s^3-27}\right]$$

$$= \underbrace{\pounds^{-1}\left[-\frac{s^2-6s-18}{(s-3)(s^2+3s+9)}\right] = \pounds^{-1}\left[\frac{1}{s-3} - \frac{2s+3}{s^2+3s+9}\right]}_{\text{By Partial Fraction Decomposition}}$$

$$= \pounds^{-1}\left[\frac{1}{s-3}\right] - \pounds^{-1}\left[\frac{2s+3}{s^2+3s+9}\right]$$

Remark 1 *(from here it looks as though our best bet for the second expression is to use "completing the square" technique on the denominator, and try to get it to fit either form 8 or form 9 on the Laplace Transform Inverses sheet.)*

$$\Rightarrow y = \pounds^{-1}\left[\mathbf{Y}(s)\right] = \pounds^{-1}\left[\frac{1}{s-3}\right] - \pounds^{-1}\left[\frac{2s+3}{s^2+3s+9}\right]$$

$$= \pounds^{-1}\left[\frac{1}{s-3}\right] - \pounds^{-1}\left[\frac{2s+3}{s^2+3s+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+9}\right] = \pounds^{-1}\left[\frac{1}{s-3}\right] - \pounds^{-1}\left[\frac{2s+3}{\left(s+\frac{3}{2}\right)^2+\frac{27}{4}}\right]$$

$$= \pounds^{-1}\left[\frac{1}{s-3}\right] - \pounds^{-1}\left[\frac{2s+3}{\left(s+\frac{3}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}\right] = \pounds^{-1}\left[\frac{1}{s-3}\right] - 2\pounds^{-1}\left[\frac{s-\left(-\frac{3}{2}\right)}{\left(s-\left(-\frac{3}{2}\right)\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}\right]$$

$$= \underbrace{e^{3t}}_{\text{form 4}} \underbrace{- 2e^{-\frac{3}{2}t}\cos\left(\frac{3\sqrt{3}}{2}t\right)}_{\text{form 9}}$$

$$\boxed{\text{i.e., } y = e^{3t} - 2e^{-\frac{3}{2}t}\cos\left(\frac{3\sqrt{3}}{2}t\right)}$$

4. $y'' - 3y' + 2y = 24\cosh(t)$; $y(0) = 6$; $y'(0) = -3$ (Note: $\cosh(t) = \frac{e^t + e^{-t}}{2}$)

(a) 1. Take the Laplace Transform of each side.

$$\Rightarrow \pounds[y'' - 3y' + 2y] = \pounds[24\cosh(t)]$$

$$\Rightarrow \pounds[y''] - 3\pounds[3y'] + 2\pounds[y] = 24\pounds[\cosh(t)]$$

$$\Rightarrow \underbrace{s^2\mathbf{Y}(s) - sy(0) - y'(0)}_{\pounds[y'']\ \text{(formula 20)}} - \underbrace{3s\mathbf{Y}(s) - y(0)}_{\pounds[y']\ \text{formula 19}} + \underbrace{2\mathbf{Y}(s)}_{\pounds[y]} = \underbrace{24\left[\frac{s}{s^2-1}\right]}_{\text{formula 11}}$$

2. Solve for $\mathbf{Y}(s)$

$$\Rightarrow (s^2 - 3s + 2)\mathbf{Y}(s) - (s-3)y(0) - y'(0) = \tfrac{24s}{s^2-1}$$

$$\Rightarrow (s^2 - 3s + 2)\mathbf{Y}(s) - (s-3)(6) - (-3) = \tfrac{24s}{s^2-1}$$

$$\Rightarrow (s^2 - 3s + 2)\mathbf{Y}(s) - 6s + 21 = \tfrac{24s}{s^2-1}$$

$$\Rightarrow (s^2 - 3s + 2)\mathbf{Y}(s) = \tfrac{24s}{s^2-1} + 6s - 21$$

$$\Rightarrow \mathbf{Y}(s) = \tfrac{24s}{(s^2-3s+2)(s^2-1)} + \tfrac{6s-21}{(s^2-3s+2)}$$

3. Solve for y, by computing the Laplace transform inverse of each side.

$$\Rightarrow y = \pounds^{-1}[\mathbf{Y}(s)] = \pounds^{-1}\left[\tfrac{24s}{(s^2-3s+2)(s^2-1)} + \tfrac{6s-21}{(s^2-3s+2)}\right]$$

$$= \pounds^{-1}\left[\tfrac{24s}{(s-1)(s-2)(s-1)(s+1)} + \tfrac{6s-21}{(s-1)(s-2)}\right] = \pounds^{-1}\left[\tfrac{24s}{(s-1)^2(s-2)(s+1)} + \tfrac{6s-21}{(s-1)(s-2)}\right]$$

$$= \pounds^{-1}\left[\tfrac{24s}{(s-1)^2(s-2)(s+1)}\right] + \pounds^{-1}\left[\tfrac{6s-21}{(s-1)(s-2)}\right]$$

$$= \underbrace{\pounds^{-1}\left[-\frac{18}{s-1} - \frac{12}{(s-1)^2} + \frac{16}{s-2} + \frac{2}{s+1}\right] + \pounds^{-1}\left[\frac{15}{s-1} - \frac{9}{s-2}\right]}_{\text{By Partial Fraction Decomposition}}$$

$$= -18\pounds^{-1}\left[\tfrac{1}{s-1}\right] - 12\pounds\left[\tfrac{1}{(s-1)^2}\right] + 16\pounds\left[\tfrac{1}{s-2}\right] + 2\pounds\left[\tfrac{1}{s+1}\right] + 15\pounds^{-1}\left[\tfrac{1}{s-1}\right]$$
$$-9\pounds\left[\tfrac{1}{s-2}\right]$$

$$= -3\pounds^{-1}\left[\tfrac{1}{s-1}\right] - 12\pounds\left[\tfrac{1}{(s-1)^2}\right] + 7\pounds\left[\tfrac{1}{s-2}\right] + 2\pounds\left[\tfrac{1}{s+1}\right]$$

$$= -3\underbrace{e^t}_{\text{form 4}} - 12\underbrace{te^t}_{\text{form 5}} + 7\underbrace{e^{2t}}_{\text{form 4}} + 2\underbrace{e^{-t}}_{\text{form 4}}$$

i.e., $y = -3e^t - 12te^t + 7e^{2t} + 2e^{-t}$