MTH 3331 Practice Test #1 - Solutions

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1. ~

(a) $A+B=\begin{bmatrix}1&2&1\\2&1&3\\0&0&1\end{bmatrix}+\begin{bmatrix}-1&1&0\\0&1&1\\1&-1&-1\end{bmatrix}=\begin{bmatrix}1+(-1)&2+1&1+0\\2+0&1+1&3+1\\0+1&0+(-1)&1+(-1)\end{bmatrix}$

$=\begin{bmatrix}0&3&1\\2&2&4\\1&-1&0\end{bmatrix}$

(b) $A^T+B^T=(A+B)^T=\begin{bmatrix}0&2&1\\3&2&-1\\1&4&0\end{bmatrix}$

(c) $AC=\begin{bmatrix}1&2&1\\2&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}1&1\\2&1\\0&4\end{bmatrix}=\begin{bmatrix}1\cdot1+2\cdot2+1\cdot0&1\cdot1+2\cdot1+1\cdot4\\2\cdot1+1\cdot2+3\cdot0&2\cdot1+1\cdot1+3\cdot4\\0\cdot1+0\cdot2+1\cdot0&0\cdot1+0\cdot1+1\cdot4\end{bmatrix}$

$=\begin{bmatrix}5&7\\4&15\\0&4\end{bmatrix}$

(d) CA is undefined. (Recall: The number of columns of the first matrix must equal the number of rows of the second matrix.)

(e) $CD=\begin{bmatrix}1&1\\2&1\\0&4\end{bmatrix}\begin{bmatrix}2&1\\1&3\end{bmatrix}=\begin{bmatrix}1\cdot2+1\cdot1&1\cdot1+1\cdot3\\2\cdot2+1\cdot1&2\cdot1+1\cdot3\\0\cdot2+4\cdot1&0\cdot1+4\cdot3\end{bmatrix}=\begin{bmatrix}3&4\\5&5\\4&12\end{bmatrix}$

(f) $BC=\begin{bmatrix}-1&1&0\\0&1&1\\1&-1&-1\end{bmatrix}\begin{bmatrix}1&1\\2&1\\0&4\end{bmatrix}$

$=\begin{bmatrix}-1\cdot1+1\cdot2+0\cdot0&(-1)\cdot1+1\cdot1+0\cdot4\\1\cdot0+1\cdot2+1\cdot0&0\cdot1+1\cdot1+1\cdot4\\1\cdot1+(-1)\cdot2+(-1)\cdot0&1\cdot1+(-1)\cdot1+(-1)\cdot4\end{bmatrix}=\begin{bmatrix}1&0\\2&5\\-1&-4\end{bmatrix}$

(g) $A+D$ is undefined. (Recall: In order to add two matrices, they must be the same size.)

(h) AD is undefined. (Recall: The number of columns of the first matrix must equal the number of rows of the second matrix.)

2. ~

   (a) $\left(A^T\right)^T = A$ Reason: To form A^T, we take the i^{th} row of A and make it the i^{th} column of A^T. (or equivalently, we take the i^{th} column of A and make it the i^{th} row of A^T.) To form $\left(A^T\right)^T$, we take the i^{th} column of A^T (which is, as you recall, the i^{th} row of A) and make it the i^{th} row of $\left(A^T\right)^T$. This means that the i^{th} row of A and the i^{th} row of $\left(A^T\right)^T$ are equal. Hence, $\left(A^T\right)^T = A$.

   (b) Recall: $(AB)^T = B^T A^T$ and $(A+B)^T = A^T + B^T$.
   Therefore, $[A(B+C)]^T = (B+C)^T A^T = \left(B^T + C^T\right) A^T = B^T A^T + C^T A^T$

3. When we transform the system of equations

$$\begin{array}{rrrcl} 4x & -7y & +3z & = & 9 \\ -2x & & +z & = & 5 \\ 8x & +3y & +2z & = & 2 \end{array}$$

to the form: $\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}$, the "coefficient matrix" A, is always such that the entries in the first column are the coefficients of x, the entries in the second column are the coefficients of y, and the entries in the third column are the coefficients of z.

Hence, $A = \begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 1 \\ 8 & 3 & 2 \end{bmatrix}$ and our system appears in matrix form as:

$$\begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 1 \\ 8 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}$$

4. Observe: For any square $(n \times n)$ matrix A,

$$\underbrace{(A-I)\left(A^2+A+I\right) = A \cdot \left(A^2+A+I\right) - I \cdot \left(A^2+A+I\right)}_{\text{distributive law}}$$

$$= A \cdot A^2 + A \cdot A + A \cdot I - I \cdot A^2 - I \cdot A - I \cdot I \text{ (distributive law again.)}$$

$$= A^3 + A^2 + A - A^2 - A - I = A^3 - I$$

5. Observe: for $(n \times n)$ matrices A and B, we have:

$$\underbrace{(A-B)\left(A^2+AB+B^2\right) = A\left(A^2+AB+B^2\right) - B\left(A^2+AB+B^2\right)}_{\text{distributive Law}} =$$

$$A \cdot A^2 + A \cdot AB + A \cdot B^2 - B \cdot A^2 + B \cdot AB + B \cdot B^2$$

(distributive law again.)

$$= A^3 + A^2B + AB^2 - BA^2 - BAB - B^3$$

Since multiplication of matrices is not commutative in general, this can't be simplified. For example, we can't rewrite BAB as AB^2 or B^2A.

6. If matrices A, B, C are such that A is nonzero and $AB = AC$, it is not necessarily true that $B = C$. For example, consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

$B \neq C$, and yet $AB = AC = \begin{bmatrix} 5 & 1 \\ 15 & 3 \end{bmatrix}$.

The reason that this can happen is that some properties of real numbers don't hold for matrices. In particular, $AB = 0_{n \times n}$ does not imply that either $A = 0$ or $B = 0$. This prevents us from showing that $AB = AC \Rightarrow B = C$. If we try to show this, here's what happens:

$AB = AC \Rightarrow \underbrace{AB - AC = 0 \Rightarrow A(B-C) = 0}_{\text{distributive law}}$. At this point,

we can't conclude that $B - C = 0$, and therefore, it doesn't follow that $B = C$.

For problems 7 to 10 we consider the **main diagonal** of an $n \times n$ matrix. The **main diagonal** is the set of elements $\{a_{11}, a_{22}, a_{33}, \ldots, a_{nn}\}$ shown below:

$$\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & a_{33} & & \\ & & & \ddots & \\ & & & & a_{nn} \end{bmatrix}$$

7. An Upper Triangular Matrix is a $(n \times n)$ matrix that has zeros for entries below the main diagonal. $\begin{bmatrix} a_{11} & & \dots & & a_{1n} \\ 0 & a_{22} & & & \\ 0 & 0 & a_{33} & & \vdots \\ \vdots & \vdots & 0 & \ddots & \\ 0 & 0 & 0 & & a_{nn} \end{bmatrix}$

8. A Lower Triangular Matrix is a $(n \times n)$ matrix that has zeros for entries above the main diagonal.

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ & a_{22} & 0 & & 0 \\ \vdots & & a_{33} & & \vdots \\ & & & \ddots & 0 \\ a_{n1} & & \dots & & a_{nn} \end{bmatrix}$$

9. A Diagonal Matrix is a $(n \times n)$ matrix that has zeros for entries above and below the main diagonal. (A Diagonal Matrix is both upper and lower triangular.)

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

10. The multiplicative identity for $n \times n$ matrices, denoted I, is the $n \times n$ matrix whose main diagonal elements are 1 and whose other elements are 0.

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

11. For

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & 6 & 3 \\ 2 & 1 & 0 \\ 9 & 8 & 5 \end{bmatrix}$$

$$4A + 2B - C = 4 \cdot \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 7 & 6 & 3 \\ 2 & 1 & 0 \\ 9 & 8 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 0 \\ 8 & 4 & 0 \\ 4 & 4 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 4 & 6 & 8 \end{bmatrix} - \begin{bmatrix} 7 & 6 & 3 \\ 2 & 1 & 0 \\ 9 & 8 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4-2-7 & 4+2-6 & 0+2-3 \\ 8+2-2 & 4-2-1 & 0+2-0 \\ 4+4-9 & 4+6-8 & 4+8-5 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -1 \\ 8 & 1 & 2 \\ -1 & 2 & 7 \end{bmatrix}$$

12. To compute the multiplicative inverse of $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$,

form the augmented matrix $[A|\ I]$.

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -\frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & -\frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 3 & -2 & 4 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & \frac{5}{9} & -\frac{4}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \end{array}\right]$$

When the left half of the augmented matrix has been transformed into the identity matrix, the right half of the augmented matrix will have been transformed into the inverse of A.

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \end{bmatrix}$$

13. ~

(a) $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

This tells us that B is the multiplicative inverse of A. (i.e. $B = A^{-1}$)

(b) $A+E = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2-1 & -1+1 & 0+0 \\ -1+1 & 2-1 & -1+1 \\ 0+0 & -1+1 & 2-1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $BA + BE = B\,(A+E) = BI = B = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

14. Given

$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$

(a) $AB = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1\cdot(-2)+2\cdot 1 & 1\cdot 4+2\cdot(-2) \\ 3\cdot(-2)+6\cdot 1 & 3\cdot 4+6\cdot(-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

i.e. $AB = 0_{2\times 2}$

(b) Problem 14.a tells us that the axiom:

$$AB = 0 \Rightarrow \text{ either } A = 0 \text{ or } B = 0$$

does not hold for matrices.

15. Simplify (i.e. State in terms of A, B, and C only):

$$\left[\left(A^T + B^T\right)\cdot C^T\right]^T = \left(C^T\right)^T \cdot \left(A^T + B^T\right)^T = C \cdot \left(\left(A^T\right)^T + \left(B^T\right)^T\right) = C\cdot(A+B)$$

$$= CA + CB$$