MTH 3331 Practice Test #2 - Solutions

Summer 2013

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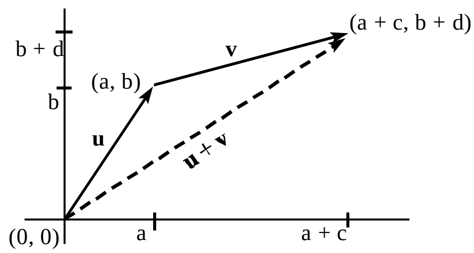
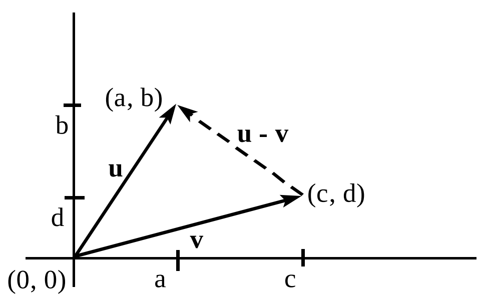
1. Given the vectors $\tilde{\mathbf{u}} = (a, b)$ and $\tilde{\mathbf{v}} = (c, d)$, give the geometric interpretation of the following:

   (a) $\tilde{\mathbf{u}} + \tilde{\mathbf{v}} = (a, b) + (c, d) = (a + c, b + d)$. Graphically, these are added in "tip to tail" fashion.



   (b) $\tilde{\mathbf{u}} - \tilde{\mathbf{v}} = (a, b) - (c, d) = (a - c, b - d)$. Graphically, these are added in "tip to tip" fashion.



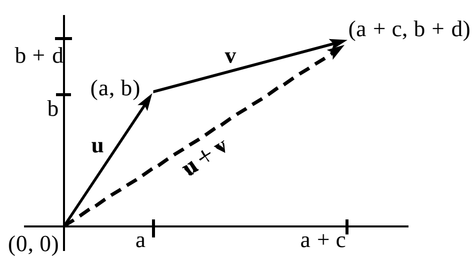
2. Find the norm or "length" of the vectors $\tilde{\mathbf{u}} = (3, 4, 12)$ and $\tilde{\mathbf{v}} = (4, 4, 5, -2)$

   $||\tilde{\mathbf{u}}|| = ||(3, 4, 12)|| = \sqrt{3^2 + 4^2 + 12^2} = 13$

   $||\tilde{\mathbf{v}}|| = ||(4, 4, 5, -2)|| = \sqrt{4^2 + 4^2 + 5^2 + (-2)^2} = \sqrt{61}$

3. **Cauchy-Schwarz Inequality**: If **u** and **v** are vectors in $\Re^n$, then $|\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}| \leq ||\tilde{\mathbf{u}}|| \, ||\tilde{\mathbf{v}}||$. Equality holds exactly when $\tilde{\mathbf{u}} = k\tilde{\mathbf{v}}$.

4. **Triangle Inequality**: If $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are vectors in $\Re^n$, then $||\tilde{\mathbf{u}} + \tilde{\mathbf{v}}|| \leq ||\tilde{\mathbf{u}}|| + ||\tilde{\mathbf{v}}||$. Equality holds exactly when $\tilde{\mathbf{u}} = k\tilde{\mathbf{v}}$. Geometrically, this means that no one side of a triangle has length greater than the sum of the lengths of the other two sides.



5. Given that $\tilde{\mathbf{u}} = (1, 2, 3, 4)$ and $\tilde{\mathbf{v}} = (3, 1, 2, 1)$, compute the dot product $\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}$.

   $\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}} = (1, 2, 3, 4) \circ (3, 1, 2, 1) = 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 1 = 15$

6. Given that $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}\right)$,

   (a) Compute $||\tilde{\mathbf{u}}||$

   $$||\tilde{\mathbf{u}}|| = \left|\left|\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}\right)\right|\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{8}}\right)^2 + \left(\frac{1}{\sqrt{8}}\right)^2} = 1$$

   (b) What does this tell us about $\tilde{\mathbf{u}}$?

   $\tilde{\mathbf{u}}$ is a unit vector.

7. Find $\cos(\theta)$, where θ is the angle between vectors $\tilde{\mathbf{u}} = (2, 4, 1)$ and $\tilde{\mathbf{v}} = (-2, 2, -4)$

   $\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}} = ||\tilde{\mathbf{u}}||\,||\tilde{\mathbf{v}}|| \cos(\theta) \Rightarrow \cos(\theta) = \frac{\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}}{||\tilde{\mathbf{u}}||||\tilde{\mathbf{v}}||} = \frac{(2,4,1)\circ(-2,2,-4)}{||(2,4,1)||||(-2,2,-4)||} = \frac{0}{\sqrt{21}\sqrt{24}} = 0.$

8. Find $\cos(\theta)$, where θ is the angle between vectors $\tilde{\mathbf{u}} = (2, 1, 4, 7)$ and $\tilde{\mathbf{v}} = (1, -1, 2, -1)$

   $\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}} = ||\tilde{\mathbf{u}}||\,||\tilde{\mathbf{v}}|| \cos(\theta) \Rightarrow \cos(\theta) = \frac{\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}}{||\tilde{\mathbf{u}}||||\tilde{\mathbf{v}}||} = \frac{(2,1,4,7)\circ(1,-1,2,-1)}{||(2,1,4,7)||||(1,-1,2,-1)||} = \frac{2}{\sqrt{70}\sqrt{7}} = \frac{2}{7\sqrt{10}} = \frac{\sqrt{10}}{35}$

9. Given the system of equations $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$, the row reduced form of the corresponding augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Give the general solution of the system.

   First, we find a particular solution. To do this, we identify the pivot and free variables:

   PIVOT

   x_1

   x_2

   FREE

   x_3

   Since the free variables can be anything, we set the free variable to zero. $\Rightarrow x_3 = 0$.

   From row 1, we get $x_1 - 2x_3 = 5 \Rightarrow x_1 = 5$

   From row 2, we get $x_2 + x_3 = -1 \Rightarrow x_2 = -1$

   $\Rightarrow x_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$ is a particular solution.

   Next, we find the general solution of the related homogeneous system. To help, we should realize that the row reduced matrix corresponding to the homogeneous system is always the same as the row reduced matrix of the original system except that the right most column is zero.

Therefore, the row reduced matrix corresponding to the homogeneous system is $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

For the general homogeneous solution, we parameterize the free variable(s).

$\Rightarrow x_3 = k$

From row 1 we have $x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3 \Rightarrow x_1 = 2k$

From row 2, we get $x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 \Rightarrow x_2 = -k$

$$\Rightarrow x_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The general solution of the non-homogeneous system is formed by adding the particular solution and the general homogeneous solution.

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{k \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}_{x_h}$$

10. Assume that each of the following matrices is the matrix of coefficients of a homogeneous system of equations. Decide whether has any solution other than the trivial solution. If so, give the solution.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ By inspection, we can see that this matrix is row equivalent to the identity matrix, and has rank3. Therefore, this system has only the trivial sollution, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Forming the augmented matrix, we have: $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

We identify the pivot variables and free variables:

PIVOT	**FREE**
x_1	x_3
x_2	

Next, we parameterize the free variable(s). $\Rightarrow x_3 = k$

Finally, we express the pivot variables in terms of the free variable(s).

From row 1, we get: $x_1 - x_3 = 0 \Rightarrow x_1 = x_3 \Rightarrow x_1 = k$

From row 2, we get $x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 \Rightarrow x_2 = -k$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(c) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Forming the augmented matrix, we have: $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

This is already row reduced.

We identify the pivot variables and free variables:

PIVOT	**FREE**
x_1	x_2
x_3	x_4

Next, we parameterize the free variable(s). $\Rightarrow x_2 = k_1$ and $x_4 = k_2$.

Finally, we express the pivot variables in terms of the free variable(s).

From row 1, we get: $x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2 \Rightarrow x_1 = -2k_1$

From row 2, we get $x_3 + x_4 = 0 \Rightarrow x_3 = -x_4 \Rightarrow x_3 = -k_2$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(d) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Forming the augmented matrix, we have:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_2 = 0$ and $x_3 = 0$. **Notice that** x_1 **is free**. Call it k. Then we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

11. If the system of equations $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ has a unique solution, what is the solution set of $A\tilde{\mathbf{x}} = \mathbf{0}$?

If the system of equations $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ has a unique solution, then the system $A\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$ has a unique solution, since it is just a special case of $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$. Since $\tilde{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a solution of $A\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$, and since the solution is unique, the system $A\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$ has only the trivial solution $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

12. Find the solution of each system.

(a) $\begin{array}{rcrcrcl} 2x & + & 3y & + & 4z & = & 2 \\ 4x & + & 5y & + & z & = & 3 \end{array}$

First, we want to get a particular solution, so we form the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 2 \\ 4 & 5 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 2 & 1 \\ 4 & 5 & 1 & 3 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 2 & 1 \\ 0 & -1 & -7 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{17}{2} & -\frac{1}{2} \\ 0 & 1 & 7 & 1 \end{array}\right]$$

Next, we identify the pivot and free variables:

PIVOT	**FREE**
x_1	x_3
x_2	

Since we're seeking **any** particular solution, and since the free variable(s) can be anything, we set the free variable(s) to zero. $\Rightarrow x_3 = 0$.

Row 1 tells us that $x_1 - \frac{17}{2}x_3 = -\frac{1}{2} \Rightarrow x_1 = -\frac{1}{2}$

Row 2 tells us that $x_2 + 7x_3 = 1 \Rightarrow x_2 = 1$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \text{ is the particular solution, } x_p.$$

Next we get the general soluton to the corresponding homogeneous system: To help, we should realize that the row reduced matrix corresponding to the homogeneous system is always the same as the row reduced matrix of the original system except that the right most column is zero.

Therefore, the row reduced matrix corresponding to the homogeneous system is:

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{17}{2} & 0 \\ 0 & 1 & 7 & 0 \end{array}\right]$$

We parameterize our free variable(s). $\Rightarrow x_3 = k$

Next, we express our pivot variables in terms of the parameter.

From row 1, we have: $x_1 - \frac{17}{2}x_3 = 0 \Rightarrow x_1 = \frac{17}{2}x_3 \Rightarrow x_1 = \frac{17}{2}k$

From row 2, we have $x_2 + 7x_3 = 0 \Rightarrow x_2 = -7x_3 \Rightarrow x_2 = -7k$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} \frac{17}{2} \\ -7 \\ 1 \end{bmatrix} \text{ This is the general homogeneous solution, } x_h.$$

The general solution of the system is the sum of a particular solution and the general homogeneous solution.

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{k \begin{bmatrix} \frac{17}{2} \\ -7 \\ 1 \end{bmatrix}}_{x_h}$$

13. Consider the system of equations $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$. Show that if $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ are solutions of the system $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$, then $(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)$ is a solution of the system $A\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$.

    If $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ are solutions of the system $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$, then $A\tilde{\mathbf{x}}_1 = \tilde{\mathbf{b}}$ and $A\tilde{\mathbf{x}}_2 = \tilde{\mathbf{b}}$.

    $\Rightarrow A\left(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\right) = A\tilde{\mathbf{x}}_1 - A\tilde{\mathbf{x}}_2 = \tilde{\mathbf{b}} - \tilde{\mathbf{b}} = \tilde{\mathbf{0}}$. (i.e., $A\left(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\right) = \tilde{\mathbf{0}} \Rightarrow \tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2$ is a solution of the corresponding homogeneous system $A\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$.

14. If $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_k$ are solutions of the system $A\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$, show that every linear combination of $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_k$ is also a solution of the system $A\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$.

    If $\tilde{\mathbf{x}}_i$ is a solution of the system $A\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$, then $A\tilde{\mathbf{x}}_i = \tilde{\mathbf{0}}$ for $i = 1, 2, \ldots, k$. A linear combination of vectors $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_k$ is an expression of the form: $c_1\tilde{\mathbf{x}}_1 + c_2\tilde{\mathbf{x}}_2 + \ldots + c_k\tilde{\mathbf{x}}_k$.

    Observe: $A\left(c_1\tilde{\mathbf{x}}_1 + c_2\tilde{\mathbf{x}}_2 + \ldots + c_k\tilde{\mathbf{x}}_k\right) = A\left(c_1\tilde{\mathbf{x}}_1\right) + A\left(c_2\tilde{\mathbf{x}}_2\right) + \ldots + A\left(c_k\tilde{\mathbf{x}}_k\right) = c_1A\tilde{\mathbf{x}}_1 + c_2A\tilde{\mathbf{x}}_2 + \ldots + c_kA\tilde{\mathbf{x}}_k = 0 + 0 + \ldots + 0 = \vec{0}$

    i.e., $c_1\tilde{\mathbf{x}}_1 + c_2\tilde{\mathbf{x}}_2 + \ldots + c_k\tilde{\mathbf{x}}_k$ is also a solution of the homogeneous system $A\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$.

15. The augmented matrix from a system of equations is given in reduced form below. Write the solution of the system as the sum of a particular solution plus the homogeneous solution.

    $$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & 0 & -1 & 6 \\ 0 & 0 & 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

    First we find the particular solution.

    Since the augmentedmatrix is already reduced, we can start off by identifying our pivot and free variables.

PIVOT	**FREE**
x_1	x_2
x_3	x_4
x_5	x_6

    Since we're seeking **any** particular solution, and since the free variable(s) can be anything, we set the free variable(s) to zero. $\Rightarrow x_2 = 0$, $x_4 = 0$, and $x_6 = 0$.

    From row 1 we have: $x_1 + 2x_2 + x_4 - x_6 = 6 \Rightarrow x_1 = 6$

    From row 2 we have: $x_3 + 4x_4 + 2x_6 = -1 \Rightarrow x_3 - 1$

    From row 3 we have: $x_5 + 2x_6 = 2 \Rightarrow x_5 = 2$

    $$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \text{ is a particular solution, } x_p$$

To find the homogeneous solution, we set the right hand side of the reduced augmented matrix equal to zero.

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since we're looking for the general homogeneous solution, we parameterize the free variables.

$\Rightarrow x_2 = k_i, \;\; x_4 = k_2, \;\; x_6 = k_3$

Next, we express the pivot variables in terms of the parameters:

From row 1 we have: $x_1 + 2x_2 + x_4 - x_6 = 0 \Rightarrow x_1 = -2x_2 - x_4 + x_6 \Rightarrow x_1 = -2k_1 - k_2 + k_3$

From row 2 we have: $x_3 + 4x_4 + 2x_6 = 0 \Rightarrow x_3 = -4x_4 - 2x_6 \Rightarrow x_3 = -4k_2 - 2k_3$

From row 3 we have: $x_5 + 2x_6 = 0 \Rightarrow x_5 = -2x_6 \Rightarrow x_5 = -2k_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{ is the homogeneous solution, } x_h.$$

The general solution of the non-homogeneous system is the sum of any particular solution plus the general homogeneous solution.

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 \\ 0 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{x_h}$$

16. If $\tilde{\mathbf{x}}_1$ is a solution of the system of equations $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$, and $\tilde{\mathbf{y}}_1$ is a solution of the system of equations $A\tilde{\mathbf{y}} = \tilde{\mathbf{0}}$, for some matrix A, what can be said about $\tilde{\mathbf{x}}_1 + \tilde{\mathbf{y}}_1$?

    If $\tilde{\mathbf{x}}_1$ is a solution of $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$, then $A \cdot \tilde{\mathbf{x}}_1 = \tilde{\mathbf{b}}$

    If $\tilde{\mathbf{y}}_1$ is a solution of the system $A\tilde{\mathbf{y}} = \tilde{\mathbf{0}}$, then $A \cdot \tilde{\mathbf{y}}_1 = \tilde{\mathbf{0}}$

    Observe: $A\left(\tilde{\mathbf{x}}_1 + \tilde{\mathbf{y}}_1\right) = A\tilde{\mathbf{x}}_1 + \mathbf{A}\tilde{\mathbf{y}}_1 = \tilde{\mathbf{b}} + \tilde{\mathbf{0}} = \tilde{\mathbf{b}}$

    (i.e., $\tilde{\mathbf{x}}_1 + \tilde{\mathbf{y}}_1$ is also a solution of the system $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$.)

17. Write the parametric equations of the line through the point $(1,4)$ and parallel to the vector $\tilde{\mathbf{v}} = (2,2)$.

    If (x,y) is a point on the line containing the point $(1,4)$ and parallel to $\tilde{\mathbf{v}}$, then $(x-1, y-4)$ is a vector containing the points (x,y) and $(1,4)$ which has the same direction as the line. (i.e., $(x-1,y-4)$ is parallel to the vector $\tilde{\mathbf{v}} = (2,2)$.)

    If $(x-1,y-4)$ is parallel to $\tilde{\mathbf{v}} = (2,2)$, then $(x-1,y-4) = k(2,2)$ for some scalar, k.

    Therefore, $\begin{array}{rl} & x-1 = 2k \\ \text{and} & y-4 = 2k \end{array}$

18. Let $p_1 = (2,2,2)$ and $p_2 = (3,5,6)$. Write the equation of the plane containing the point p_1 and normal to the vector p_1p_2 (i.e normal to the vector with endpoints p_1 and p_2.)

    $p_1p_1 = (3-2, 5-2, 6-2) = (1,3,4)$ is a normal vector to the plane.

    If (x,y,z) is any point in the plane, then $(x-2, y-2, z-2)$ is a vector in the plane.

    Since vector $(1,3,4)$ is orthogonal to the plane, $(x-2,y-2,z-2) \circ (1,3,4) = \tilde{\mathbf{0}}$.

    $\Rightarrow x - 2 + 3y - 6 + 4z - 8 = 0 \;\Rightarrow\; x + 3y + 4z = 16.$

19. Write the equation of the plane in $\mathbf{R}^3$ which contains the point $(1,1,1)$, and is parallel to the plane given by the equation $2x + 2y - 3z = 16$.

    Two parallel lines have the same normal vectors. A normal vector for the plane

$$\begin{array}{ccccccc} 2x & + & 2y & -3z & = & 16 & \text{is} \\ \searrow & & \downarrow & \downarrow & & & \\ & (2, & 2, & -3) & & & \end{array}$$

    This is also a normal vector to the plane we seek.

    If (x,y,z) is any point in the parallel plane, then $(x-1,y-1,z-1)$ is a vector in the plane. Furthermore, $(x-1,y-1,z-1) \circ (2,2,-3) = 0$.

    $\Rightarrow 2x - 2 + 2y - 2 - 3z + 3 = 0 \Rightarrow 2x + 2y - 3z = 1$

20. Write the parametric equations for the line which contains the point $(2,1,2)$ and which is parallel to the vector $\tilde{\mathbf{v}} = (-1,3,3)$.

    If (x,y,z) is a point on the line containing the point $(2,1,2)$ and parallel to $\tilde{\mathbf{v}}$, then $(x-2,y-1,z-2)$ is a vector containing the points (x,y,z) and $(2,1,2)$ which has the same direction as the line. (i.e., $(x-2,y-1,z-2)$ is parallel to the vector $\tilde{\mathbf{v}} = (-1,3,3)$)

    If $(x-2,y-1,z-2)$ is parallel to $\tilde{\mathbf{v}} = (-1,3,3)$, then $(x-2,y-1,z-2) = k(-1,3,3)$ for some scalar, k.

    Therefore, $\begin{array}{rl} & x-2 = -k \\ & y-1 = 3k \\ \text{and} & z-2 = 3k \end{array}$

21. Compute A^{-1} if $A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Form the augmented matrix $[A \mid I]$ and transform it to: $[I \mid A]$.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

(replace row 2 with the sum of row 2 + (-1)row 1)

(replace row 3 with the sum of row 3 + (-2)row 1)

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -1 & 1 & 0 \\ 0 & -8 & 5 & -2 & 0 & 1 \end{array}\right]$$

Multiply row 2 by $-\frac{1}{3}$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -8 & 5 & -2 & 0 & 1 \end{array}\right]$$

Replace row 3 with the sum of row 3 + (8)row

2

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{8}{3} & 1 \end{array}\right]$$

Multiply row 3 by -3

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -2 & 8 & -3 \end{array}\right]$$

(replace row 2 with the sum of row 2 + $\left(\frac{2}{3}\right)$row 3)

(replace row 1 with the sum of row 1 + row 1)

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & -1 & 8 & -3 \\ 0 & 1 & 0 & -1 & 5 & -2 \\ 0 & 0 & 1 & -2 & 8 & -3 \end{array}\right]$$

replace row 1 with the sum of row 1 + (-4)row 2

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -12 & 5 \\ 0 & 1 & 0 & -1 & 5 & -2 \\ 0 & 0 & 1 & -2 & 8 & -3 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 3 & -12 & 5 \\ -1 & 5 & -2 \\ -2 & 8 & -3 \end{bmatrix}$$

22. Use the result of problem #21 to solve the system:

$$\begin{array}{rcrcrcr} x & + & 4y & - & z & = & 4 \\ x & + & y & + & z & = & 2 \\ 2x & & & + & 3z & = & -1 \end{array}$$

This system has the form: $[A]\,\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ where A is the matrix of the previous problem.

We will have solved the system if we find $\tilde{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

But note that $[A]\,\tilde{\mathbf{x}} = \tilde{\mathbf{b}} \Rightarrow \underbrace{[A^{-1}]\,[A]}_{I}\tilde{\mathbf{x}} = [A^{-1}]\,\tilde{\mathbf{b}} \Rightarrow \tilde{\mathbf{x}} = [A^{-1}]\,\tilde{\mathbf{b}}$

$$\Rightarrow \tilde{\mathbf{x}} = \begin{bmatrix} 3 & -12 & 5 \\ -1 & 5 & -2 \\ -2 & 8 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \tilde{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -17 \\ 8 \\ 11 \end{bmatrix}$$

23. Let $p_1 = (1,1,1)$ and $p_2 = (3,4,5)$. Write the equation of the plane which contains the point p_1 and is normal to the vector p_1p_2 (i.e normal to the vector with endpoints p_1 and p_2.)

$p_1p_1 = (3-1, 4-1, 5-1) = (2,3,4)$ is a normal vector to the plane.

If (x,y,z) is any point in the plane, then $(x-1, y-1, z-1)$ is a vector in the plane.

Since vector $(2,3,4)$ is orthogonal to the plane, $(x-1, y-1, z-1) \circ (2,3,4) = \tilde{\mathbf{0}}$.

$\Rightarrow 2x - 2 + 3y - 3 + 4z - 4 = 0 \;\Rightarrow\; 2x + 3y + 4z = 9.$

24. Write the equation of the plane in $\mathbf{R}^3$ which contains the point $(2,1,2)$, and is parallel to the plane given by the equation $4x - 4y + 3z = 20$.

Two parallel lines have the same normal vectors. A normal vector for the plane

$$\begin{array}{cccccccc} 4x & -4y & + & 3z & = & 20 & \text{is} \\ \downarrow & \downarrow & & \swarrow & & & \\ (4, & -4, & 3) & & & & \end{array}$$

This is also a normal vector to the plane we seek.

If (x,y,z) is any point in the parallel plane, then $(x-2, y-1, z-2)$ is a vector in the plane. Furthermore, $(x-2, y-1, z-2) \circ (4,-4,3) = 0$.

$\Rightarrow 4x - 8 - 4y + 4 + 3z - 6 = 0 \Rightarrow 4x - 4y + 3z = 10$

25. In each case below, the augmented matrix of a system of equations is given in row reduced form. Write the solution of the system as the sum of $n \times 1$ arrays. Do this by finding a particular solution and finding the homogeneous solution, and then adding the two together.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ same as $\begin{array}{ccccccc} x_1 & & & + & x_3 & = & 2 \\ & & x_2 & + & 2x_3 & = & 3 \end{array}$

x_1 and x_2 are the "pivot variables" and x_3 is a "free variable".

Step 1 Get a particular solution by setting the free variables equal to zero.

$$\Rightarrow \begin{array}{l} x_1 = 2 \\ x_2 = 3 \\ x_3 = 0 \end{array} \quad \text{or} \quad \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}}_{\text{particular solution}}$$

Step 2 Get a homogeneous solution:

For the homogeneous system, the reduced augmented matrix is the same as in step #1, except that the right most column is zero.

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ same as $\begin{array}{ccccccc} x_1 & & & + & x_3 & = & 0 \\ & & x_2 & + & 2x_3 & = & 0 \end{array}$

x_3 is a "free variable", so we'll parameterize it. Let $x_3 = k$. Then:

$x_1 = -x_3 = -k$
$x_2 = -2x_3 = -2k$
$x_3 = k$

or $\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}}_{\text{homogeneous solution}}$

recall: general solution = particular solution + homogeneous solution

$$\Rightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{gen sol}} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}}_{\text{part sol}} + \underbrace{k \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}}_{\text{hom sol}}$$

(b) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$ same as $\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & & & & & = & 5 \\ & & & & x_3 & & & = & 3 \\ & & & & & & x_4 & = & 1 \end{array}$

x_1, x_3 and x_4 are the "pivot variables" and x_2 is a "free variable".

Step 1 Get a particular solution by setting the free variables equal to zero.

$\Rightarrow$ $x_1 = 5$, $x_2 = 0$, $x_3 = 3$, $x_4 = 1$ or $\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \\ 1 \end{bmatrix}}_{\text{particular solution}}$

Step 2 Get a homogeneous solution:

For the homogeneous system, the reduced augmented matrix is the same as in step #1, except that the right most column is zero. (Always!)

$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ same as $\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & & & & & = & 0 \\ & & & & x_3 & & & = & 0 \\ & & & & & & x_4 & = & 0 \end{array}$

x_2 is a "free variable", so we'll parameterize it. Let $x_2 = k$. Then:

$x_1 = -2x_2 = -2k$
$x_2 = k$
$x_3 = 0$
$x_4 = 0$

or $\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = k \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{homogeneous solution}}$

recall: general solution = particular solution + homogeneous solution

$$\Rightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\text{gen sol}} = \underbrace{\begin{bmatrix} 5 \\ 0 \\ 3 \\ 1 \end{bmatrix}}_{\text{part sol}} + \underbrace{k \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{hom sol}}$$

(c) $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ same as $\begin{array}{cccccccccc} x_1 & + & 2x_2 & & & & + & 4x_5 & = & 0 \\ & & & x_3 & & & + & 2x_5 & = & 0 \\ & & & & & x_4 & + & 3x_5 & = & 0 \end{array}$

x_1, x_3 and x_4 are the "pivot variables" and x_2 and x_5 are the "free variables".

Step 1 Since this is a homogeneous system of equations, the general solution IS the homogeneous solution. Parameterize the free variables x_2 and x_5.

Let $x_2 = k_1$ and $x_2 = k_{2.}$ Then:

$x_1 = -2x_2 - 4x_5 = -2k_1 - 4k_2$

$x_2 = k_1$

$x_3 = -2x_5 = -2k_2$

$x_4 = -3x_5 = -3k_2$

$x_5 = k_2$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -4 \\ 0 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$