MTH #3331 Practice Test #3 - Solutions

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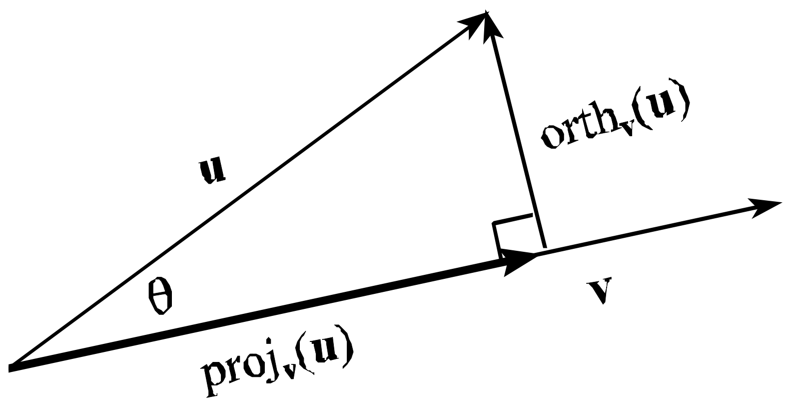
1. Given vectors $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (1, 1)$, compute:

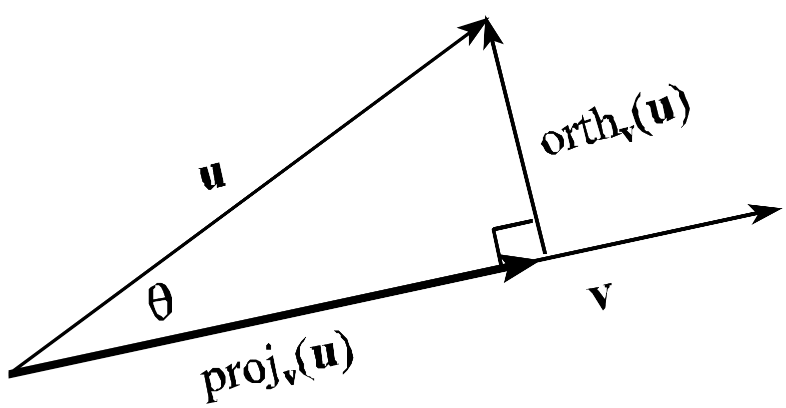
   (a) $\mathbf{proj_v}(\mathbf{u})$



From the picture, we can see that the magnitude of $\mathbf{proj_v}(\mathbf{u})$ is $\mathbf{u} \cdot \cos(\theta)$. Recall that $\mathbf{u} \circ \mathbf{v} = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos(\theta) \Rightarrow \cos(\theta) = \frac{\mathbf{u} \circ \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||}$. It follows that the magnitude of $\mathbf{proj_v}(\mathbf{u})$ is given by $||\mathbf{u}|| \cdot \cos(\theta) = ||\mathbf{u}|| \cdot \frac{\mathbf{u} \circ \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||}$. To get a vector with this magnitude and the direction of $\mathbf{v}$, we multiply this by the normalized version of $\mathbf{v}$, namely $\frac{\mathbf{v}}{||\mathbf{v}||}$. We have: $\mathbf{proj_v}(\mathbf{u}) = \underbrace{||\mathbf{u}|| \cdot \frac{\mathbf{u} \circ \mathbf{v}}{||\mathbf{u}|| \, ||\mathbf{v}||}}_{\text{magnitude}} \cdot \underbrace{\frac{\mathbf{v}}{||\mathbf{v}||}}_{\text{length}} = \frac{\mathbf{u} \circ \mathbf{v}}{||\mathbf{v}||^2} \cdot \mathbf{v} = \frac{(2,3) \circ (1,1)}{\left(\sqrt{1^2+1^2}\right)^2} \cdot$

$(1, 1) = \frac{5}{2} \cdot (1, 1) = \left(\frac{5}{2}, \frac{5}{2}\right)$.
i.e., $\mathbf{proj_v}(\mathbf{u}) = \left(\frac{5}{2}, \frac{5}{2}\right)$

   (b) $\mathbf{orth_v}(\mathbf{u})$



Notice from the picture, that $\mathbf{proj_v}(\mathbf{u}) + \mathbf{orth_v}(\mathbf{u}) = \mathbf{u}$. This implies that $\mathbf{orth_v}(\mathbf{u}) = \mathbf{u} - \mathbf{proj_v}(\mathbf{u}) = (2, 3) - \left(\frac{5}{2}, \frac{5}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$.
i.e., $\mathbf{orth_v}(\mathbf{u}) = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

2. Calculate $\det(A)$ for:

(a) $A = \begin{bmatrix} 1 & -1 & 2 & -4 \\ 1 & 0 & -4 & 2 \\ 1 & -1 & 0 & 0 \\ 2 & 2 & -2 & 6 \end{bmatrix}$

Since A is bigger than 3×3, we have to compute the determinant using cofactors. The trick is to make our computation easier by expanding along the row or column that has the most zeros. Expanding along row 3, we have:

$$\begin{aligned}
\det(A) &= a_{31}\gamma_{31} + a_{32}\gamma_{32} + a_{33}\gamma_{33} + a_{34}\gamma_{34} \\
&= 1 \cdot (-1)^{3+1} \det(A_{31}) + (-1) \cdot (-1)^{3+2} \det(A_{32}) + 0 \cdot (-1)^{3+3} \det(A_{33}) \\
&\quad +0 \cdot (-1)^{3+4} \det(A_{34}) \\
&= \det\left(\begin{bmatrix} -1 & 2 & -4 \\ 0 & -4 & 2 \\ 2 & -2 & 6 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 2 & -4 \\ 1 & -4 & 2 \\ 2 & -2 & 6 \end{bmatrix}\right) \\
&= [(24 + 8 + 0) - (32 + 4 + 0)] + [(-24 + 8 + 8) - (32 - 4 + 12)] \\
&= -52
\end{aligned}$$

(b) $A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & -4 \\ 2 & 1 & 0 & -4 & 2 \\ 3 & 1 & -1 & 0 & 0 \\ 4 & 2 & 2 & -2 & 6 \end{bmatrix}$

To make computation easy, we expand along the first row, since it has the most zeros. We have:

$$\begin{aligned}
\det(A) &= a_{11}\gamma_{11} + a_{12}\gamma_{12} + a_{13}\gamma_{13} + a_{14}\gamma_{14} + a_{15}\gamma_{15} \\
&= 3 \cdot \gamma_{11} + 0 \cdot \gamma_{12} + 0 \cdot \gamma_{13} + 0 \cdot \gamma_{14} + 0 \cdot \gamma_{15} \\
&= 3 \cdot \det\left(\begin{bmatrix} 1 & -1 & 2 & -4 \\ 1 & 0 & -4 & 2 \\ 1 & -1 & 0 & 0 \\ 2 & 2 & -2 & 6 \end{bmatrix}\right) = 3 \cdot \underbrace{(-52)}_{\text{from part a}} = -156
\end{aligned}$$

3. Calculate $\det(A)$ for:

(a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Since row 2 is a multiple of row 1 (row 2 $= 0\cdot$ row 1), $\det(A) = 0$.

(b) $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 5 & 7 \end{bmatrix}$

Since row 2 is a multiple of row 1 (row 2 $= 0\cdot$ row 1), $\det(A) = 0$.

(c) $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}$

Recall, $\det(A) = \det\left(A^T\right) = \det\left(\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0$, because row 3 is a multiple of row 1 (row 3 $= 0\cdot$ row 1).

(d) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Since row 2 is a multiple of row 1 (row 2 $= 2\cdot$ row 1), $\det(A) = 0$.

(e) $A = \begin{bmatrix} 3 & 1 & 4 \\ 3 & 1 & 4 \\ 1 & 5 & 7 \end{bmatrix}$

Since row 2 is a multiple of row 1 (row 2 $= 1\cdot$ row 1), $\det(A) = 0$.

(f) $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 2 & -4 & 5 \end{bmatrix}$

Since row 3 is a linear combination of rows 1 and 2 (row 3 = row 1 + row 2), $\det(A) = 0$.

Remark 1 *Notice that in all six cases above, the rows are linearly* ***de****pendent, hence* A *is singular and* $\det(A) = 0$.

4. If A and B are matrices, and $\det(A) = \det(B)$, is it necessarily true that $A = B$?

No. Consider, for example, $A = \begin{bmatrix} 2 & 4 \\ 4 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

Computation will show that $\det(A) = 8$ and $\det(B) = 8$, and yet $A \neq B$.

5. Write as a sum of two determinants, and compute:

(a) $$\det\left(\begin{bmatrix} 1-k & 2+0 & 3+0 \\ 4 & 1 & 7 \\ 2 & 8 & 9 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 7 \\ 2 & 8 & 9 \end{bmatrix}\right) + \det\left(\begin{bmatrix} -k & 0 & 0 \\ 4 & 1 & 7 \\ 2 & 8 & 9 \end{bmatrix}\right)$$

$$= [(9 + 28 + 96) - (6 + 56 + 72)] + [(-9k) - (-56k)] = -1 + 47k$$

(b) $$\det\left(\begin{bmatrix} 1-k & 4-2 & 1+2 \\ 4 & 1 & 7 \\ 2 & 8 & 9 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 4 & 1 \\ 4 & 1 & 7 \\ 2 & 8 & 9 \end{bmatrix}\right) + \det\left(\begin{bmatrix} -k & -2 & 2 \\ 4 & 1 & 7 \\ 2 & 8 & 9 \end{bmatrix}\right)$$

$$= [(9 + 56 + 32) - (2 + 56 + 144)] + [(-9k - 28 + 64) - (4 - 56k - 72)] = -1 + 47k$$

6. Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

To make this as easy as possible, we expand along the row or column with the most zeros, so we'll expand along row 3. We have:

$$\det(A) = a_{31}\gamma_{31} + a_{32}\gamma_{32} + a_{33}\gamma_{33} + a_{34}\gamma_{34}$$

$$= 1 \cdot \gamma_{31} + 0 \cdot \gamma_{32} + 0 \cdot \gamma_{33} + 0 \cdot \gamma_{34}$$

$$= (-1)^{3+1} \det\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) = 3.$$

7. Compute the determinants using cofactors:

(a) $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

To make this as easy as possible, we expand along the row or column with the most zeros, so we'll expand along row 2. We have:

$$\det(A) = a_{21}\gamma_{21} + a_{22}\gamma_{22} + a_{23}\gamma_{23} = 0 \cdot \gamma_{21} + 1 \cdot \gamma_{22} + 0 \cdot \gamma_{23}$$

$$= (-1)^{2+2} \det\left(\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\right) = 0.$$

(b) $A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 2 & 1 & 5 & 2 \\ 3 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 \end{bmatrix}$

To make this as easy as possible, we expand along the row or column with the most zeros, so we'll expand along column 2. We have:

$$\det(A) = a_{12}\gamma_{12} + a_{22}\gamma_{22} + a_{32}\gamma_{32} + a_{42}\gamma_{42} = 0 \cdot \gamma_{12} + 1 \cdot \gamma_{22} + 0 \cdot \gamma_{32} + 0 \cdot \gamma_{42}$$

$$= (-1)^{2+2} \det\left(\begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & 1 \\ 4 & 0 & 1 \end{bmatrix}\right) = [(0 + 12 + 0) - (0 + 0 + 9)] = 3$$

(c) $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 1 & 4 & 2 \\ 0 & 5 & 0 & 7 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

Row 1 has 2 zeros, so let's expand about it. We have:

$$\det(A) = \det(A) = a_{11}\gamma_{11} + a_{12}\gamma_{12} + a_{13}\gamma_{13} + a_{14}\gamma_{14}$$

$$= 1 \cdot \gamma_{11} + 0 \cdot \gamma_{12} + 2 \cdot \gamma_{13} + 0 \cdot \gamma_{14}$$

$$= (-1)^{1+1} \det\left(\begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 7 \\ -1 & 1 & -1 \end{bmatrix}\right) + 2 \cdot (-1)^{1+3} \det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 7 \\ 1 & -1 & -1 \end{bmatrix}\right)$$

$$= -5 + 2(3) = 1$$

8. Combine the method of row reduction and cofactors to calculate $\det(A)$, if:

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & 4 & 2 \\ 1 & -1 & 5 & 7 \\ -2 & 1 & 1 & 3 \end{bmatrix}$$

Recall, that the row operation of replacing one row with the sum of that row plus a constant multiple of another row, does not change the determinant. So let's reduce the matrix so that the determinant will be easier to compute.

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & 4 & 2 \\ 1 & -1 & 5 & 7 \\ -2 & 1 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & 2 & 6 \\ 0 & 3 & 7 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Since the determinant of a triangular matrix is the product of the diagonal elements, $\det(A) = -24$.

9. Calculate $\det(A)$, $\det(B)$, and $\det(AB)$ if:

$$A = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 2 & 0 & -1 & 3 \\ 1 & 2 & 0 & 2 \\ -1 & 1 & 0 & -1 \end{bmatrix}$$

$$\det(A) = a_{12}\gamma_{12} + a_{22}\gamma_{22} + a_{32}\gamma_{32} + a_{42}\gamma_{42} = 0 \cdot \gamma_{12} + 0 \cdot \gamma_{22} + 2 \cdot \gamma_{32} + 1 \cdot \gamma_{42}$$

$$= 2 \cdot (-1)^{3+2} \det\left(\begin{bmatrix} 1 & 1 & 5 \\ 2 & -1 & 3 \\ -1 & 0 & -1 \end{bmatrix} \right) + 1 \cdot (-1)^{4+2} \det\left(\begin{bmatrix} 1 & 1 & 5 \\ 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \right)$$

$$= -2 \cdot [(1 - 3 + 0) - (5 + 0 - 2)] + [(-2 + 3 + 0) - (-5 + 0 + 4)] = 12$$

$$B = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Let's do some row reduction first.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Since this is an upper triangular matrix, the determinant is the product of the diagonal elements. Therefore, $\det(B) = 6$.

Now recall that $\det(AB) = \det(A)\det(B) = 12 \cdot 6 = 72$.

10. Characterize the following matrices as singular or non-singular. Justify your answers.

(a) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

Singular. The rows are linearly dependent (row2 = 2 row1).

(b) $\begin{bmatrix} 1 & 2 & 1 \\ -3 & 0 & 6 \\ -2 & 2 & 7 \end{bmatrix}$

Singular. The rows are linearly dependent (row3 = row1 + row2).

(c) $\begin{bmatrix} 1 & 2 & 4 & 1 \\ 7 & 0 & 0 & 0 \\ 6 & -1 & 1 & 0 \\ 11 & 0 & 0 & 4 \end{bmatrix}$

$$\det(A) = a_{21}\gamma_{21} = 7 \cdot (-1)^{2+1} \det\left(\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right) = -7 \cdot 24 \neq 0.$$

Since the determinant is nonzero, the matrix is non-singular.

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 6 & 7 & 1 \end{bmatrix}$

Since A is a lower triangular matrix, the determinant is just the product of the diagonal elements. Therefore, $\det(A) = 2$.

Since $\det(A) \neq 0$, A is non-singular.

11. Which of the following matrices is singular?

(a) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Since A is an upper triangular matrix, the determinant is just the product of the diagonal elements. Therefore, $\det(A) = 6 \neq 0$.

Since $\det(A) \neq 0$, A is non-singular.

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}$

Since the rows are linearly dependent $(row3 = 3 \cdot row1)$, $\det(A) = 0$. Hence, and A is singular.

(c) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Non-singular, because $\det(A) = 4 \neq 0$.

(d) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Singular, because the rows are linearly dependent (row3 = row1 + row2), and therefore $\det(A) = 0$.

12. Given that A is a 6×6 matrix, and that $\det(A) = 5$, what is the rank of A?

Rank$(A) = 6$.

Since $\det(A) \neq 0$, A is non-singular, and hence A^{-1} exists. Since A^{-1} exists, A is row equivalent to the identity, and therefore, rank$(A) = n = 6$.

13. The system of equations $A\mathbf{x} = \mathbf{b}$ is a system of five equations and five unknowns. It has a unique solution. What is the rank of A?

Rank$(A) = 5$.

Since the system $A\mathbf{x} = \mathbf{b}$ has a unique solution, rank$(A) = n = 5$.

14. A is the matrix $\begin{bmatrix} 1 & a & 0 & 0 \\ a & 1 & 0 & 0 \\ 1 & a & 1 & a \\ 1 & -1 & a & 1 \end{bmatrix}$. Show that if the system of equations $A\mathbf{x} = \mathbf{0}$ has more than one solution, then we must have: $a = 1$ or $a = -1$.

Recall: the system of equations $A\mathbf{x} = \mathbf{0}$ has exactly one solution if A^{-1} exists, which, in turn implies that $\det(A) \neq 0$. So in order for $A\mathbf{x} = \mathbf{0}$ to have more than one solution, we must have $\det(A) = 0$.

To compute the determinant, we'll eliminate **above** the main diagonal (this is because the determinant of a triangular matrix is the product of the diagonals.

$$\begin{bmatrix} 1 & a & 0 & 0 \\ a & 1 & 0 & 0 \\ 1 & a & 1 & a \\ 1 & -1 & a & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & a & 0 & 0 \\ a & 1 & 0 & 0 \\ 1-a & 2a & 1-a^2 & 0 \\ 1 & -1 & a & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-a^2 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 1-a & 2a & 1-a^2 & 0 \\ 1 & -1 & a & 1 \end{bmatrix}$$

Since the matrix is now lower triangular, the determinant is the product of the diagonals. We have: $\det(A) = (1-a^2)^2 = 0$. This yields $a = \pm 1$.

15. If A and B are $n \times n$ matrices with rank n, show that AB has rank n.

If A and B are $n \times n$ matrices with rank n, then $\det(A) \neq 0$ and $\det(B) \neq 0$. $\Rightarrow \det(AB) = \det(A) \cdot \det(B) \neq 0$. Since $\det(AB) \neq 0$, the rank of AB is n.

16. If A and B are $n \times n$ matrices with rank less than n, show that AB has rank less than n.

If A and B are $n \times n$ matrices with rank less than n, then $\det(A) = 0$ and $\det(B) = 0$. $\Rightarrow \det(AB) = \det(A) \cdot \det(B) = 0$. Since $\det(AB) = 0$, the rank of AB is less than n.

17. For what value(s) of λ does $A\mathbf{x} = \lambda\mathbf{x}$ have a nontrivial solution, if $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$?

Find the vectors $\mathbf{x}$ associated with each λ.

This is equivalent to asking: "What are the eigenvalues and eigenvectors of

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$?"

To find the eigenvalues, we set the determinant of $(A - \lambda I) = 0$, and solve for λ.

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1, 3.$$

For the eigenvalue $\lambda_1 = 1$, we find the eigenvectors by solving the system $(A - \lambda I)\,\mathbf{x} = 0$.

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_2 \text{ is free} \Rightarrow x_2 = k \Rightarrow x_1 = -k.$$

The eigenvector is the solution of the system, $\Rightarrow k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigenvector associated

with $\lambda_1 = 1$.

For the eigenvalue $\lambda_2 = 3$, we find the eigenvectors by solving the system $(A - \lambda I)\,\mathbf{x} = 0$.

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_2 \text{ is free} \Rightarrow x_2 = k \Rightarrow x_1 = k.$$

The eigenvector is the solution of the system, $\Rightarrow k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector associated

with $\lambda_2 = 3$.

18. Find the eigenvalues and eigenvectors of A:

(a) $\begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$

To find the eigenvalues, we set the determinant of $(A - \lambda I) = 0$, and solve for λ.

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 4 & 6-\lambda \end{bmatrix}\right) = (3-\lambda)(6-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda-2)(\lambda-7) = 0$$

$\Rightarrow \lambda = 2, 7$ are eigenvalues.

For the eigenvalue $\lambda_1 = 2$, we find the eigenvectors by solving the system

$(A - \lambda I)\,\mathbf{x} = 0.$

$$\begin{bmatrix} 3-\lambda & 1 \\ 4 & 6-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 3-\lambda & 1 & 0 \\ 4 & 6-\lambda & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Observe that x_1 is a pivot variable and x_2 is a free variable.

Parameterizing, we have $x_2 = k$ and $x_1 = -k$. This means that $k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigenvector associated with $\lambda_1 = 2$.

For the eigenvalue $\lambda_2 = 7$, we find the eigenvectors by solving the system

$(A - \lambda I)\,\mathbf{x} = 0.$

$$\begin{bmatrix} 3-\lambda & 1 \\ 4 & 6-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 3-\lambda & 1 & 0 \\ 4 & 6-\lambda & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -4 & 1 & 0 \\ 4 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 4 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Observe that x_1 is a pivot variable and x_2 is a free variable.

Parameterizing, we have, $x_2 = k$ and $x_1 = \frac{1}{4}k$. This means that $k\begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$ is the eigenvector associated with $\lambda_2 = 7$.

(b) $\begin{bmatrix} 7 & 7 & 7 \\ -5 & -7 & -9 \\ 5 & 7 & 9 \end{bmatrix}$

To find the eigenvalues, set $\det(A - \lambda I) = 0$, and solve for λ.

$$\det\left(\begin{bmatrix} 7-\lambda & 7 & 7 \\ -5 & -7-\lambda & -9 \\ 5 & 7 & 9-\lambda \end{bmatrix}\right) = -\lambda^3 + 9\lambda^2 - 14\lambda = 0$$

$\Rightarrow \lambda(\lambda - 2)(\lambda - 7) = 0 \Rightarrow \lambda = 0, 2, 7.$

For the eigenvalue $\lambda_1 = 0$, we find the eigenvectors by solving the system

$(A - \lambda I)\,\mathbf{x} = 0.$

$$\begin{bmatrix} 7-\lambda & 7 & 7 \\ -5 & -7-\lambda & -9 \\ 5 & 7 & 9-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 7-\lambda & 7 & 7 & 0 \\ -5 & -7-\lambda & -9 & 0 \\ 5 & 7 & 9-\lambda & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 7 & 7 & 7 & 0 \\ -5 & -7 & -9 & 0 \\ 5 & 7 & 9 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -5 & -7 & -9 & 0 \\ 5 & 7 & 9 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Observe that x_1 and x_2 are pivot variables and x_3 is a free variable.

Parameterizing, we have: $x_3 = k;\ \Rightarrow\ x_1 = k;\ \Rightarrow\ x_2 = -2k$

Therefore, the eigenvector corresponding to $\lambda_1 = 0$ is $k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

For the eigenvalue $\lambda_2 = 2$, we find the eigenvectors by solving the system

$(A - \lambda I)\,\mathbf{x} = 0$.

$$\begin{bmatrix} 7-\lambda & 7 & 7 \\ -5 & -7-\lambda & -9 \\ 5 & 7 & 9-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 7-\lambda & 7 & 7 & 0 \\ -5 & -7-\lambda & -9 & 0 \\ 5 & 7 & 9-\lambda & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 5 & 7 & 7 & 0 \\ -5 & -9 & -9 & 0 \\ 5 & 7 & 7 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{7}{5} & \frac{7}{5} & 0 \\ -5 & -9 & -9 & 0 \\ 5 & 7 & 7 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{7}{5} & \frac{7}{5} & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{7}{5} & \frac{7}{5} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Observe that x_1 and x_2 are pivot variables and x_3 is a free variable.

Parameterizing, we have: $x_3 = k;\ x_1 = k;\ x_2 = -k$.

Therefore, the eigenvector corresponding to $\lambda_2 = 2$ is $k \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

For the eigenvalue $\lambda_3 = 7$, we find the eigenvectors by solving the system

$(A - \lambda I)\,\mathbf{x} = 0.$

$$\begin{bmatrix} 7-\lambda & 7 & 7 \\ -5 & -7-\lambda & -9 \\ 5 & 7 & 9-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 7-\lambda & 7 & 7 & 0 \\ -5 & -7-\lambda & -9 & 0 \\ 5 & 7 & 9-\lambda & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 7 & 7 & 0 \\ -5 & -14 & -9 & 0 \\ 5 & 7 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 5 & 7 & 2 & 0 \\ -5 & -14 & -9 & 0 \\ 0 & 7 & 7 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{7}{5} & \frac{2}{5} & 0 \\ -5 & -14 & -9 & 0 \\ 0 & 7 & 7 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{7}{5} & \frac{2}{5} & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 7 & 7 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{7}{5} & \frac{2}{5} & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 7 & 7 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{7}{5} & \frac{2}{5} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 7 & 7 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{7}{5} & \frac{2}{5} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Observe that x_1 and x_2 are pivot variables and x_3 is a free variable.

Parameterizing, we have $x_3 = k;\ x_1 = k;\ x_2 = -k.$

Therefore, the eigenvector corresponding to $\lambda_3 = 7$ is $k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 3 & -1 & 2 \\ 0 & -1 & 0 \\ -4 & 2 & -3 \end{bmatrix}$

Looking for eigenvalues, we set the determinant of $(A - \lambda I) = 0$.

$$\det\left(\begin{bmatrix} 3-\lambda & -1 & 2 \\ 0 & -1-\lambda & 0 \\ -4 & 2 & -3-\lambda \end{bmatrix}\right) = -\lambda^3 - \lambda^2 + \lambda + 1 = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - \lambda - 1 = 0 \Rightarrow (\lambda - 1)(\lambda + 1)^2 = 0$$

$\Rightarrow \lambda_1 = 1;\ \lambda_2 = -1;\ \lambda_3 = -1$ are the eigenvalues.

To find the eigenvector corresponding to $\lambda_1 = 1$, we solve the system

$$\left[\begin{array}{ccc|c} 3-\lambda & -1 & 2 & 0 \\ 0 & -1-\lambda & 0 & 0 \\ -4 & 2 & -3-\lambda & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ -4 & 2 & -4 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 1 & 0 \\ 0 & -2 & 0 & 0 \\ -4 & 2 & -4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 1 & 0 \\ 0 & -2 & 0 & 0 \\ -0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Observe that x_1 and x_2 are pivot variables and x_3 is a free variable.

Parameterizing, we have: $x_3 = k;\ x_1 = -k;\ x_2 = 0$.

Therefore, the eigenvector associated with $\lambda_1 = 1$ is $k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

To find the eigenvector(s) corresponding to $\lambda_2 = \lambda_3 = -1$, we solve the system

$$\left[\begin{array}{ccc|c} 3-\lambda & -1 & 2 & 0 \\ 0 & -1-\lambda & 0 & 0 \\ -4 & 2 & -3-\lambda & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 4 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 2 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 2 & -2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & & 0 & 0 \end{array}\right]$$

Observe that x_1 and x_2 are pivot variables and x_3 is a free variable.

Parameterizing, we have: $x_3 = k$; $x_1 = -\frac{1}{2}k$; $x_2 = 0$.

Therefore the eigenvector that is associated with $\lambda_2 = \lambda_3 = -1$ is $k \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$.

19. Given that $\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, state one eigenvalue of the matrix $\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 3 & 2 & 0 \end{bmatrix}$, and its corresponding eigenvector.

If λ is an eigenvalue of a given a matrix A, then there exists a vector $\mathbf{x}$, such that $A\mathbf{x} = \lambda\mathbf{x}$.

Since $\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, it follows that $\lambda = 5$ is an eigenvalue, and

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ or more generally, $\begin{bmatrix} k \\ k \\ k \end{bmatrix}$, is the corresponding eigenvector.

20. Calculation will show that $\det\left(\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}\right) = 0$. What does this information tell us about the eigenvalues of the matrix?

If $\det(A) = 0$, then $\det(A) = \det(A - 0I) = 0$. This means that $\lambda = 0$ is an eigenvalue.