MTH #3331 Practice Test #4 - Solutions

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1. Complete the following statement: The following statements are equivalent for an $n \times n$ matrix A :

   (a)
   1. A is non-singular.
   2. A is row equivalent to I.
   3. A has rank n.
   4. A is invertible (i.e., A^{-1} exists).
   5. The system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution.
   6. The system of equations $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
   7. The rows of A are linearly independent.
   8. The columns of A are linearly independent.
   9. $\det(A) \neq 0$.
   10. The rows of A span $\Re^n$.
   11. The columns of A span $\Re^n$.

2. Complete the following statement: The following statements are equivalent for an $n \times n$ matrix A :

   (a)
   1. A is singular.
   2. A is **not** row equivalent to I.
   3. A has rank less than n.
   4. A is **not** invertible (i.e., A^{-1} **does not** exist).
   5. The system of equations $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.
   6. The system of equations $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.
   7. The rows of A are linearly dependent.
   8. The columns of A are linearly dependent.
   9. $\det(A) = 0$.
   10. The rows of A **do not** span $\Re^n$.
   11. The columns of A **do not** span $\Re^n$.

3. Define a **basis** of $\Re^n$.

   A **basis** of $\Re^n$ is a set of vectors of $\Re^n$ which is linearly independent, and spans of $\Re^n$.

   Alternately, a **basis** of $\Re^n$ is a set of vectors of $\Re^n$ which has the property that every vector in $\Re^n$ can be expressed **uniquely** as a linear combination of the elements of the set.

4. Define what it means for a set of n-tuples to **span** $\Re^n$.

   A set of n-tuples of $\Re^n$ is said to **span** $\Re^n$ if every vector in $\Re^n$ can be written as a linear combination of the n-tuples of the set.

5. What can be said about:

   (a) a set of n-tuples in $\Re^n$ having less than n elements?

      1. Since the set does not have exactly n elements, the set cannot be a basis of $\Re^n$.

      2. Since the set has less than n elements, it fails to span $\Re^n$.

   (b) a set of n-tuples in $\Re^n$ having more than n elements?

      1. Since the set does not have exactly n elements, the set cannot be a basis of $\Re^n$.

      2. Since the set has more than n elements, the elements of the set are linearly dependent.

6. How can a set of n-tuples fail to be a basis of $\Re^n$?

   A set of n-tuples can fail to be a basis of $\Re^n$ by either

   (a) not spanning of $\Re^n$, or

   (b) not being linearly independent (or both).

7. Given the set of 3-tuples $\{(1,1,0),(0,1,1),(1,0,1),(1,1,1)\}$, show that the fourth 3-tuple is a linear combination of the first three.

   If the fourth 3-tuple is a linear combination of the first three, then there are scalars c_1, c_2, c_3 such that $c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_3 \\ c_1 + c_2 \\ c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   Row 1 tells us that $c_1 + c_3 = 1 \Rightarrow c_3 = 1 - c_1$

   Row 2 tells us that $c_1 + c_2 = 1 \Rightarrow c_2 = 1 - c_1$

   Row 3 tells us that $c_2 + c_3 = 1 \Rightarrow (1 - c_1) + (1 - c_1) = 1 \Rightarrow -2c_1 + 2 = 1$

   $\Rightarrow c_1 = \frac{1}{2} \Rightarrow c_2 = \frac{1}{2} \Rightarrow c_3 = \frac{1}{2}$

   i.e. $\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

8. Decide by inspection whether each set is linearly independent or dependent. Give a reason for your answer.

(a) $\{(1,1,2),(1,4,5),(1,2,7),(-1,8,3)\}$ Linearly dependent.

Any set of more than 3 vectors in $\Re^3$ must be linearly dependent.

(b) $\{(1,1,0,0),(0,0,1,1)\}$ Linearly independent

$$\text{If } c_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},$$

$$\text{then}\quad c_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}c_1\\c_1\\c_2\\c_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$\Rightarrow c_1=0$ and $c_2=0$. Therefore, the vectors are linearly independent.

(c) $\{(1,0,0,0),(1,1,0,0),(1,1,1,0)\}$ Linearly independent

$$\text{If } c_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+c_3\begin{bmatrix}1\\1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},$$

$$\text{then}\quad c_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+c_3\begin{bmatrix}1\\1\\1\\0\end{bmatrix}=\begin{bmatrix}c_1+c_2+c_3\\c_2+c_3\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Row 3 tells us that $c_3=0\Rightarrow$(by row 2) $c_2=0\Rightarrow$(by row 1) $c_1=0$.

Since $c_1=0$; $c_2=0$; and $c_3=0$; the vectors are linearly independent.

(d) The columns of the matrix $\begin{bmatrix}1&5&4\\2&8&-3\end{bmatrix}$ Linearly dependent.

The columns of the matrix are elements of $\Re^2$. Since more than 2 vectors in $\Re^2$, are always dependent, the columns are linearly dependent.

(e) $\{(0,0,0),(1,1,5),(2,8,7)\}$ Linearly dependent

There is a linear combination of the vectors $c_1(0,0,0)+c_2(1,1,5)+c_3(2,8,7)=(0,0,0)$ such that not all of the c_i 's are zero. For example $1\cdot(0,0,0)+0\cdot(1,1,5)+0\cdot(2,8,7)=(0,0,0)$. Therefore, the vectors are linearly dependent.

9. Show that any set containing the zero vector, $\mathbf{0}$, is linearly dependent.

   Let $\{\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a collection of vectors in $\Re^n$. Then $1 \cdot \mathbf{0} + 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + \ldots + 0 \cdot \mathbf{v}_k = \mathbf{0}$ is a non trivial linear combination of the vectors which equals 0. Therefore, the vectors are linearly dependent.

10. Show that if the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent, then any subset of this set is also linearly independent.

    Let the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be linearly independent, and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_j\}$ be a subset $(j < k)$.

    If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_j\}$ **were** linearly dependent, then there would be scalars $c_1, c_2, \ldots, c_j$, not all zero, such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_j\mathbf{u}_j = 0$.

    But if this were the case, then we could let $c_{j+1} = 0;\ c_{j+2} = 0;\ \ldots\ c_k = 0$, in which case we'd have:

    $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_j\mathbf{u}_j + c_{j+1}\mathbf{u}_{j+1} + \ c_{j+2}\mathbf{u}_{j+2} + \ c_k\mathbf{u}_k = \mathbf{0}$.

    This is a non trivial linear combination (not all c_i 's are zero) that's equal to $\mathbf{0}$, which implies that the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly dependent, a contradiction.

    Since assuming that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_j\}$ was dependent led to a contradiction, this assumption must be false.

    Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_j\}$ is independent, and in general, any subset of a linearly independent set is linearly independent.

11. Show that if the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly dependent, then any set containing these as a subset is also linearly dependent.

    Suppose that the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly dependent.

    Then there are scalars $c_1, c_2, \ldots, c_k$, not all zero, such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k = \mathbf{0}$.

    Suppose that S is a set containing $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.

    Without loss of generality, we can represent S as $\left\{ \underbrace{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k}_{\text{elements of original set}}, \mathbf{u}_{k+1}, \mathbf{u}_{k+2}, \ldots, \mathbf{u}_n \right\}$.

    We can choose the same $c_1, c_2, \ldots, c_k$ as before and let $c_{k+1} = 0;\ c_{k+2} = 0;\ \ldots\ ; c_n = 0$.

    We have:

    $$\underbrace{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k}_{=0 \text{ from above}} + \underbrace{c_{k+1}\mathbf{u}_{k+1}, c_{k+2}\mathbf{u}_{k+2}, \ldots, c_n\mathbf{u}_n}_{=0 \text{ because } c_i=0 \text{ for } i>k} = \mathbf{0}, \text{ with not all } c_i = 0.$$

    Therefore, S is linearly dependent.

12. By inspection, determine whether A^{-1} exists for each of the following. (Give a reason for your answer.)

(a) $\begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ A^{-1} does not exist.

By definition, a **two sided** inverse (left and right) can exist only if A is square.

(b) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$ A^{-1} exists.

The matrix is triangular, which implies that the determinant is the product of the diagonals.

Since $\det(A) = 56 \neq 0$, the inverse exists.

(c) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ A^{-1} does not exist.

Row 2 = Row 1. Since the rows are linearly dependent, A^{-1} does not exist.

(d) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ A^{-1} does not exist, because A is not square.

(e) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ A^{-1} does not exist.

Row 2 = $2 \cdot$ row 1. Since the rows are linearly dependent, A^{-1} does not exist.

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ A^{-1} exists

Elimination above the main diagonal will yield: $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{5}{3} & 0 \\ 2 & 1 & 3 \end{bmatrix}$.

From here it is obvious that the original matrix is row equivalent to the identity matrix, I, and therefore, A^{-1} exists.

13. Let $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for $\Re^n$. What can be said about the following? (Give more of a reason for your answer than $\mathbf{B}'$ is a basis or $\mathbf{B}'$ is not a basis.)

   (a) $\mathbf{B}' = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n-1}\}$

   $\mathbf{B}'$ is not a basis of $\Re^n$. Since $\mathbf{B}'$ is a subset of the linearly independent set $\mathbf{B}$, $\mathbf{B}'$ is linearly independent also. Finally, since $\mathbf{B}'$ has less than n elements, $\mathbf{B}'$ does not span $\Re^n$.

   (b) $\mathbf{B}' = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n-2}, \mathbf{x}\}$ where $\mathbf{x}$ is a vector in $\Re^n$.

   $\mathbf{B}'$ is not a basis of $\Re^n$. Since $\mathbf{B}'$ has less than n elements, it cannot span $\Re^n$. $\mathbf{B}'$ may or may not be linearly independent, depending on what $\mathbf{x}$ is.

   (c) $\mathbf{B}' = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n-1}, \mathbf{b}_n, \mathbf{x}\}$ where $\mathbf{x}$ is a vector in $\Re^n$.

   $\mathbf{B}'$ is not a basis of $\Re^n$. Since $\mathbf{B}'$ has more than n elements, it is linearly dependent. Since $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ spans $\Re^n$, $\mathbf{B}' = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n-1}, \mathbf{b}_n, \mathbf{x}\}$ must span $\Re^n$ also.

14. Let $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ be a set of vectors in $\Re^n$. Suppose also that there are elements $\mathbf{s}_1$ and $\mathbf{s}_2$ in $\Re^n$ such that $\mathbf{s}_1 = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_m\mathbf{b}_m$ and $\mathbf{s}_2 = k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + \ldots + k_m\mathbf{b}_m$ with scalars $c_i \neq k_i$ for $i = 1, 2, \ldots, n$. What can we say about $\mathbf{B}$, and why, if $\mathbf{s}_1 - \mathbf{s}_2 = 0$?

    $\mathbf{B}$ is not a basis of $\Re^n$. The element $\mathbf{s}_1$ is an element that cannot be expressed **uniquely** as a linear combination of the elements of $\mathbf{B}$, for $\mathbf{s}_1 = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_m\mathbf{b}_m = k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + \ldots + k_m\mathbf{b}_m = \mathbf{s}_2$.

    Alternatively, $\mathbf{s}_1 - \mathbf{s}_2 = 0 \Rightarrow c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_m\mathbf{b}_m - (k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + \ldots + k_m\mathbf{b}_m) = 0$

    $\Rightarrow (c_1 - k_1)\,\mathbf{b}_1 + (c_2 - k_2)\,\mathbf{b}_2 + \ldots + (c_m - k_m)\,\mathbf{b}_m = \mathbf{0}$ with $c_i \neq k_i$ for $i = 1, 2, \ldots, m$.

    This implies that $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ is linearly dependent.

15. Which of the following could be a basis for $\Re^3$? Explain why or why not (e.g., if you claim linear independence or dependence, give a reason why we know the set is linearly independent or dependent.)

(a) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Yes. Viewing the vectors as columns of a matrix, row reduction yields:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\Rightarrow$ The matrix is row equivalent to the identity, I.

Therefore, the columns are linearly independent.

Any 3 linearly independent vectors in $\Re^3$ also span $\Re^3$ and hence, constitute a basis.

(b) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

No. The vectors are not linearly independent. (The third column is the sum of the first two.)

(c) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

No. Two vectors can't span $\Re^3$.

(d) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

Yes. If we view the vectors as being the columns of a matrix, the matrix is triangular, with $\det(A) = 1$ (The product of the diagonals). Since $\det(A) \neq 0$, the columns are linearly independent. This implies that they also span $\Re^3$ (see problem 1).

(e) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Yes. Same reason as previous problem.

(f) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

No. A colection of more than 3 elements in $\Re^3$ is linearly dependent.

(g) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ -4 \\ 3 \end{bmatrix} \right\}$

No. The vectors are dependent. Viewing the vectors as columns of a matrix, we can see that row 3 = row 1 + row 2. This means that the rows are dependent. This implies that the columns are dependent.

(h) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

No. The vectors are not linearly independent, as the third vector is the sum of the first two.

16. Determine, by inspection, which of the following sets of 4-tuples are bases for $\Re^4$. Give a reason in each case.

(a) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Not a basis. Any collection of less than 4 vectors can't span $\Re^4$.

(b) $\mathbf{B} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}$

This is a basis. Viewing the vectors as the columns of a matrix, the matrix is triangular, with the determinant being the product of the diagonal elements. therefore, $\det(A) = 80 \neq 0$. This tells us that the columns are linearly independent, by problem 1. Also by problem 1, the columns span $\Re^4$.

(c) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 6 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 9 \\ 6 \end{bmatrix} \right\}$

Not a basis. Viewing the vectors as the columns of a matrix, we can see that the rows are not linearly independent, as row 4 = $2 \cdot$ row 1. Since the rows are dependent, the columns are dependent, also.

(d) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Not a basis. Any collection of more than 4 vectors in $\Re^4$ is dependent.

(e) $\mathbf{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Not a basis. The fourth element is the sum of the first and third. Therefore, the columns are dependent.

17. Write in row reduced (row echelon) form, and determine the rank:

$$A = \begin{bmatrix} 1 & 4 & 3 & 6 \\ 2 & 9 & 8 & 7 \\ 6 & 26 & 22 & 26 \\ 5 & 22 & 19 & 20 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{replace row 2 with sum of row 2 } +(-2)\text{row1} \\ \text{replace row 3 with sum of row 3 } +(-6)\text{ row 1} \\ \text{replace row 4 with sum of row4 } +(-5)\text{ row1}}]{} \Rightarrow \begin{bmatrix} 1 & 4 & 3 & 6 \\ 0 & 1 & 2 & -5 \\ 0 & 2 & 4 & -10 \\ 0 & 2 & 4 & -10 \end{bmatrix}$$

$$\xrightarrow[\substack{\text{replace row 3 wiht the sum of row 3 } +(-2)\text{row2} \\ \text{replace row 4 with the sum of row 4 } +(-2)\text{row2}}]{} \Rightarrow \begin{bmatrix} 1 & 4 & 3 & 6 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{replace row 1 with the sum of row1 } +(-4)\text{ row 2} \Rightarrow \begin{bmatrix} 1 & 0 & -5 & 26 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank is the number of non-zero rows after elimination, so rank is 2.

18. Solve the system of equations by transforming $\left[A\,|\,\vec{b}\right] \Rightarrow \left[I|\,\vec{b'}\right]$.

$$\left[\begin{array}{ccccc} x_1 & +2x_2 & +x_3 & +x_4 & = 2 \\ x_1 & -x_2 & +4x_3 & -4x_4 & = -4 \\ 2x_1 & +x_2 & +5x_3 & +5x_4 & = -2 \\ 3x_1 & & -9x_3 & -9x_4 & = -6 \end{array}\right]$$

Form the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 1 & -1 & 4 & -4 & -4 \\ 2 & 1 & 5 & 5 & -2 \\ 3 & 0 & -9 & -9 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 0 & -3 & 3 & -5 & -6 \\ 0 & -3 & 3 & 3 & -6 \\ 0 & -6 & -12 & -12 & -12 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & \frac{5}{3} & 2 \\ 0 & -3 & 3 & 3 & -6 \\ 0 & -6 & -12 & -12 & -12 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & \frac{5}{3} & 2 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & -18 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & \frac{5}{3} & 2 \\ 0 & 0 & -18 & -2 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & \frac{5}{3} & 2 \\ 0 & 0 & 1 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{c} -2 \\ 2 \\ 0 \\ 0 \end{array}\right]$$

19. Let $\vec{u}_1 = (1, 2, 1, 3)$; $\vec{u}_2 = (1, -1, 2, 0)$; $\vec{u}_3 = (1, a, 2, b)$. Choose a and b so that the set $\{\vec{u}_1,\ \vec{u}_2,\ \vec{u}_3\}$ is linearly dependent.

Construct a matrix whose rows are $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$. If the vectors are dependent, then forward elimination will make the bottom row zero.

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & -1 & 2 & 0 \\ 1 & a & 2 & b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & 1 & -3 \\ 0 & a-2 & 1 & b-3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & a-2 & 1 & b-3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 + \frac{1}{3}(a-2) & (b-3)-(a-2) \end{bmatrix}$$

If the vectors are linearly dependent, then the bottom row will be the zero vector.

This means that $1 + \frac{1}{3}(a-2) = 0 \Rightarrow \frac{1}{3} + \frac{1}{3}a = 0 \Rightarrow a = -1$

Also: $(b-3)-(a-2) = 0 \Rightarrow b = a+1 \Rightarrow b = 0$

So $a = -1$ and $b = 0$.