MTH 3331 Test #1 - Solutions

Summer 2015

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Instructions. Show clearly how you arrive at your answers.

1. Write the system of equations

$$\begin{array}{rrrcl} 4x & -7y & +3z & = & 9 \\ -2x & & +z & = & 5 \\ 8x & +3y & +2z & = & 2 \end{array}$$

in the form: $\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}$

The coefficients of x, y, z are the entries of the "coefficient matrix," A. This yields:

$$\begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 1 \\ 8 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}$$

2. For $(n \times n)$ matrices A and B, compute $(A - B)(A^2 + AB + B^2)$, simplifying to the extent possible.

To do this, we have to use the distributive laws:

$$\begin{aligned} (A - B)(A^2 + AB + B^2) &= A(A^2 + AB + B^2) - B(A^2 + AB + B^2) \\ &= AA^2 + AAB + AB^2 - BA^2 + BAB + BB^2 \\ &= A^3 + A^2B + AB^2 - BA^2 - BAB - B^3 \end{aligned}$$

Since matrix multiplication isn't commutative, This can't be simplified.

Alternatively:

$$\begin{aligned} (A - B)(A^2 + AB + B^2) &= (A - B)A^2 + (A - B)AB + (A - B)B^2 \\ &= AA^2 - BA^2 + AAB - BAB + AB^2 - BB^2 \\ &= A^3 - BA^2 + A^2B - BAB + AB^2 - B^3 \end{aligned}$$

Again, since matrix multiplication isn't commutative, This can't be simplified.

3. If matrices A, B, C are such that A is nonzero and $AB = AC$, what can we conclude?

   Based on this information alone, we can conclude nothing. The left and right "Cancelation Laws" do not hold for matrix multiplication, so we CAN'T conclude that $B = C$.

4. Define Diagonal Matrix.

   A **diagonal matrix** is a matrix that is both upper and lower triangular.

   Alternately, a **diagonal matrix** is a matrix whose only non-zero elements lie on the main diagonal (i.e., $a_{ij} = 0$ if $i \neq j$.)

5. What is the multiplicative identity for $n \times n$ matrices?

   The multiplicative identity, denoted I (or $I_{n\times n}$), is the $n \times n$ matrix whose entries are given by the following:

$$a_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ \\ 0 & \text{for} \quad i \neq j \end{cases}$$

   It has the property that for every $n \times n$ matrix, A,

$$AI = A = IA$$

6. For

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & 6 & 3 \\ 2 & 1 & 0 \\ 9 & 8 & 5 \end{bmatrix}$$

   compute $4A + 2B - C$.

$$\begin{aligned} 4A + 2B - C &= 4\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + 2\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 7 & 6 & 3 \\ 2 & 1 & 0 \\ 9 & 8 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 4 & 0 \\ 8 & 4 & 0 \\ 4 & 4 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 4 & 6 & 8 \end{bmatrix} - \begin{bmatrix} 7 & 6 & 3 \\ 2 & 1 & 0 \\ 9 & 8 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 0 & -1 \\ 8 & 1 & 2 \\ -1 & 2 & 7 \end{bmatrix} \end{aligned}$$

7. Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

(a) Compute AB

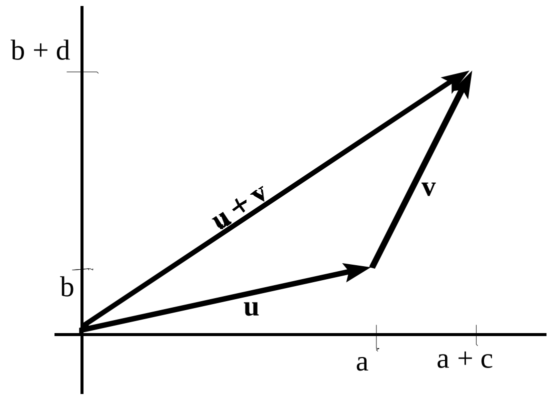
$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) What does the result in problem 7.a tell us about matrix multiplication?

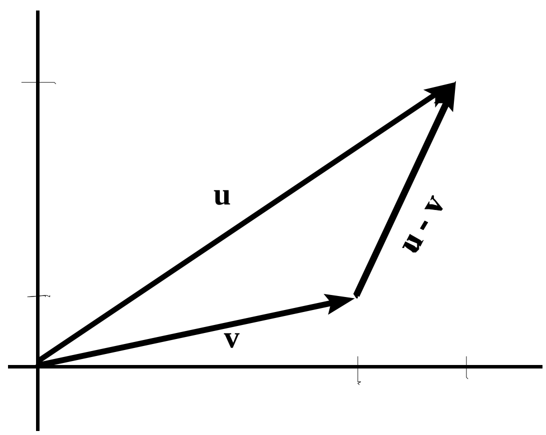
If the product of two matrices is zero (i.e., the zero matrix) we CAN'T conclude that one of the matrices is necessarily zero (i.e., the zero matrix).

8. Given the vectors $\tilde{\mathbf{u}} = (a, b)$ and $\tilde{\mathbf{v}} = (c, d)$, give the geometric interpretation of the following:

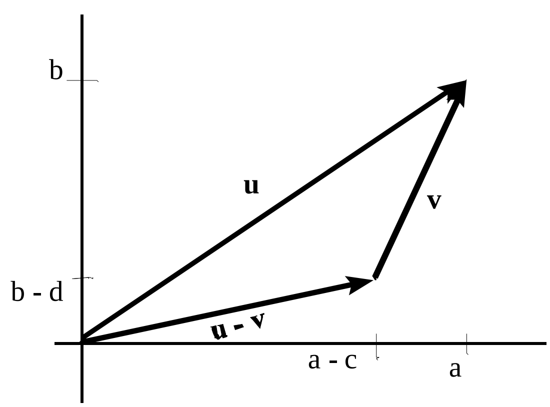
(a) $\tilde{\mathbf{u}} + \tilde{\mathbf{v}}$



(b) $\tilde{\mathbf{u}} - \tilde{\mathbf{v}}$



ALTERNATIVELY:



9. For the matrices given below, calculate each of the following products when they are defined.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

(a) AB

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

(b) AC

$$AC = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 4 & 15 \\ 0 & 4 \end{bmatrix}$$

(c) CD

$$CD = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 5 \\ 4 & 12 \end{bmatrix}$$

(d) BC

$$BC = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ -1 & -4 \end{bmatrix}$$

(e) DC

$$DC = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix}}_{3\times 2} \text{ this is undefined } 2 \times \underbrace{2 \text{ and } 3}_{\text{not the same}} \times 2$$

(f) $BC + CD$

$$BC + CD = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ -1 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 5 & 5 \\ 4 & 12 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 7 & 10 \\ 3 & 8 \end{bmatrix}$$

(g) $(A+B)\,C$

$$(A+B)\,C = AC + BC = \begin{bmatrix} 6 & 7 \\ 6 & 20 \\ -1 & 0 \end{bmatrix} \quad \text{(From the next part)}$$

(h) $AC + BC$

$$AC + BC = \begin{bmatrix} 5 & 7 \\ 4 & 15 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 6 & 20 \\ -1 & 0 \end{bmatrix}$$

10. State the Cauchy-Schwarz Inequality.

For vectors, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ in $\mathbf{R}^n$,

$$|\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}| \leq ||\tilde{\mathbf{u}}||\,||\tilde{\mathbf{v}}||$$

with equality exactly when $\tilde{\mathbf{u}} = k\tilde{\mathbf{v}}$, for any scalar, k.

11. State the Triangle Inequality and give its geometric interpretation.

For vectors, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ in $\mathbf{R}^n$,

$$||\tilde{\mathbf{u}} + \tilde{\mathbf{v}}|| \leq ||\tilde{\mathbf{u}}|| + ||\tilde{\mathbf{v}}||$$

with equality exactly when $\tilde{\mathbf{u}} = k\tilde{\mathbf{v}}$, for any scalar, $k > 0$.

This means that no side of a triangle has greater length that the sums of the lengths of the other two sides.

12. Given that $\tilde{\mathbf{u}} = (1,2,3,4)$ and $\tilde{\mathbf{v}} = (3,1,2,1)$, compute the dot product $\mathbf{u} \circ \mathbf{v}$.

$\mathbf{u} \circ \mathbf{v} = (1,2,3,4) \circ (3,1,2,1) = (3+2+6+4) = 15$

13. Find $\cos(\theta)$, where θ is the angle between vectors $\tilde{\mathbf{u}} = (2,4,1)$ and $\tilde{\mathbf{v}} = (-2,2,-4)$

$\cos(\theta) = \frac{\mathbf{u} \circ \mathbf{v}}{||\tilde{\mathbf{u}}||\,||\tilde{\mathbf{v}}||} = \frac{(2,4,1)\circ(-2,2,-4)}{||(2,4,1)||\,||(-2,2,-4)||} = \frac{0}{\sqrt{21}\sqrt{24}} = 0$

i.e., $\cos(\theta) = 0$

14. Find the norm or "length" of the vectors $\tilde{\mathbf{u}} = (3, 4, 12)$ and $\tilde{\mathbf{v}} = (4, 4, 5, -2)$

$$||\tilde{\mathbf{u}}|| = ||(3, 4, 12)|| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$$

$$||\tilde{\mathbf{v}}|| = ||(4, 4, 5, -2)|| = \sqrt{4^2 + 4^2 + 5^2 + (-2)^2} = \sqrt{61}$$

15. For $n \times n$ matrices A, B, C, write the transpose of each matrix

(a) A^T

$$\left(A^T\right)^T = A$$

(b) AA^T

$$\left[AA^T\right]^T = \left(A^T\right)^T (A)^T = AA^T$$

(c) $A(B + C)$

$$[A(B + C)]^T = (B + C)^T A^T = \left(B^T + C^T\right) A^T = B^T A^T + C^T A^T$$

(d) ABC

$$[ABC]^T = C^T B^T A^T$$

(e) $A^T BA$

$$\left[A^T BA\right]^T = A^T B^T \left(A^T\right)^T = A^T B^T A$$

(f) $A^T B + B^T A$

$$\left[A^T B + B^T A\right]^T = \left(A^T B\right)^T + \left(B^T A\right)^T = B^T \left(A^T\right)^T + A^T \left(B^T\right)^T = B^T A + A^T B$$

16. Show that $||k\mathbf{v}|| = |k|\,||\mathbf{v}||$. (Hint: Somewhere along the line, you'll have to use the fact that the absolute value of a product equals the product of the absolute values. i.e., $|ab| = |a|\,|b|$.

Observe: $||k\mathbf{v}|| = ||k(v_1, v_2, \ldots, v_n)|| = ||(kv_1, kv_2, \ldots, kv_n)||$

$$= \sqrt{(kv_1)^2 + (kv_2)^2 + \ldots + (kv_n)^2} = \sqrt{k^2 v_1^2 + k^2 v_2^2 + \ldots + k^2 v_n^2}$$

$$= \sqrt{k^2\left(v_1^2 + v_2^2 + \ldots + v_n^2\right)} = \sqrt{k^2}\sqrt{\left(v_1^2 + v_2^2 + \ldots + v_n^2\right)}$$

$$= |k|\,||(v_1, v_2, \ldots, v_n)|| = |k|\,||\mathbf{v}||$$

17. ~

(a) Find a unit vector that has the same direction as the vector $(-1,4,3)$.

To form such a vector, we divide this vector by its length.

$\tilde{\mathbf{u}} = \frac{(-1,4,3)}{||(-1,4,3)||} = \frac{(-1,4,3)}{\sqrt{1+16+9}} = \left(-\frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{3}{\sqrt{26}}\right) = \left(\frac{\sqrt{26}}{26}, \frac{2\sqrt{26}}{13}, \frac{3\sqrt{26}}{26}\right)$

(b) Find a unit vector that has the opposite direction of the vector $(0,2,5)$.

This is essentially the same process, except that we "reverse the direction" by multiplying the unit vector by -1.

$-\tilde{\mathbf{u}} = -\frac{(0,2,5)}{||(0,2,5)||} = -\frac{(0,2,5)}{\sqrt{0^2+2^2+5^2}} = -\frac{(0,2,5)}{\sqrt{29}} = \left(0, -\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right) = \left(0, -\frac{2\sqrt{29}}{29}, -\frac{5\sqrt{29}}{29}\right)$

18. Verify the Cauchy-Schwartz Inequality in the following case:

(a) $\mathbf{u} = (1,-1,3,4)$ and $\mathbf{v} = (2,0,3,1)$.

$|\mathbf{u} \circ \mathbf{v}| = |(1,-1,3,4) \circ (2,0,3,1)| = |15| = \sqrt{225} \leq \sqrt{378}$

$= \sqrt{27}\sqrt{14} = ||(1,-1,3,4)||\,||(2,0,3,1)|| = ||\mathbf{u}||\,||\mathbf{v}||$

19. Let $\mathbf{u} = (1,1,2)$ and $\mathbf{v} = (k,0,3)$. Choose k such that:

(a) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\mathbf{u} \circ \mathbf{v} = 0$

So we want, $\mathbf{u} \circ \mathbf{v} = (1,1,2) \circ (k,0,3) = k + 6 = 0$

$\Rightarrow k = -6$

(b) the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{6}$.

If the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{6}$, then $\cos(\theta) = \frac{\sqrt{3}}{2}$.

So we want $\frac{\mathbf{u} \circ \mathbf{v}}{||\tilde{\mathbf{u}}||||\tilde{\mathbf{v}}||} = \frac{(1,1,2) \circ (k,0,3)}{||(1,1,2)||||(k,0,3)||} = \frac{k+6}{\sqrt{6}\sqrt{k^2+9}} = \frac{\sqrt{3}}{2}$.

$\frac{k+6}{\sqrt{6}\sqrt{k^2+9}} = \frac{\sqrt{3}}{2} \Rightarrow \frac{k^2+12k+36}{6(k^2+9)} = \frac{3}{4} \Rightarrow 4k^2 + 48k + 144 = 18k^2 + 162$

$\Rightarrow 14k^2 - 48k + 18 = 0 \Rightarrow 7k^2 - 24k + 9 = 0 \Rightarrow (7k-3)(k-3) = 0$

$\Rightarrow k = \frac{3}{7}, 3$