MTH 3331 Test #1 - Solutions

Spring 2018

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Instructions. Show clearly how you arrive at your answers.

1. Write the system of equations

$$\begin{array}{rcrcrcl} 2x & + & 3y & - & 8z & = & 3 \\ -2x & + & y & + & 4z & = & 7 \\ 8x & + & 3y & + & 2z & = & 2 \end{array}$$

in the form: $\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ \vec{x} \end{bmatrix} = \begin{bmatrix} \\ \vec{b} \end{bmatrix}$

The coefficients of x, y, z are the entries of the "coeffiecient matrix," A. This yields:

$$\begin{bmatrix} 2 & 3 & -8 \\ -2 & 1 & 4 \\ 8 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$$

2. For $(n \times n)$ matrices A and B, compute $(A - B)(A^2 + AB + B^2)$, simplifying to the extent possible.

To do this, we have to use the distributive laws:

$$\begin{aligned} (A - B)(A^2 + AB + B^2) &= A(A^2 + AB + B^2) - B(A^2 + AB + B^2) \\ &= AA^2 + AAB + AB^2 - BA^2 + BAB + BB^2 \\ &= A^3 + A^2B + AB^2 - BA^2 - BAB - B^3 \end{aligned}$$

Since matrix multiplication isn't commutative, This can't be simplified.

Alternatively:

$$\begin{aligned} (A - B)(A^2 + AB + B^2) &= (A - B)A^2 + (A - B)AB + (A - B)B^2 \\ &= AA^2 - BA^2 + AAB - BAB + AB^2 - BB^2 \\ &= A^3 - BA^2 + A^2B - BAB + AB^2 - B^3 \end{aligned}$$

Again, since matrix multiplication isn't commutative, This can't be simplified.

3. If matrices A, B, C are such that A is nonzero and $AB = AC$, what can we conclude?

   Based on this information alone, we can conclude nothing. The left and right "Cancelation Laws" do not hold for matrix multiplication, so we CAN'T conclude that $B = C$.

4. If matrices A and B are $n \times n$ matrices, and $AB = 0_{n\times n}$, what can we conclude?

   Based on this information alone, we can conclude nothing. The "Zero Factor Property" does not hold true for matrix multiplication. (i.e., if A and B are $n \times n$ matrices, then $AB = 0_{n\times n}$ does NOT imply that $A = 0_{n\times n}$ or $B = 0_{n\times n}$)

5. Define Diagonal Matrix.

   A **diagonal matrix** is a matrix that is both upper and lower triangular.

   Alternatively, a **diagonal matrix** is a matrix whose only non-zero elements lie on the main diagonal (i.e., $a_{ij} = 0$ if $i \neq j$.)

6. What is the multiplicative identity for $n \times n$ matrices?

   The multiplicative identity, denoted I (or $I_{n\times n}$), is the $n \times n$ matrix whose entries are given by the following:

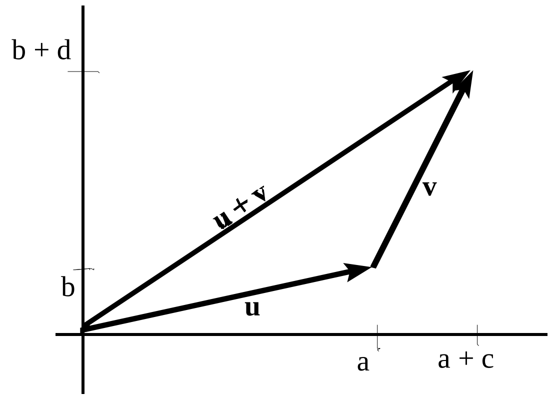
$$a_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ \\ 0 & \text{for} \quad i \neq j \end{cases}$$

   It has the property that for every $n \times n$ matrix, A,

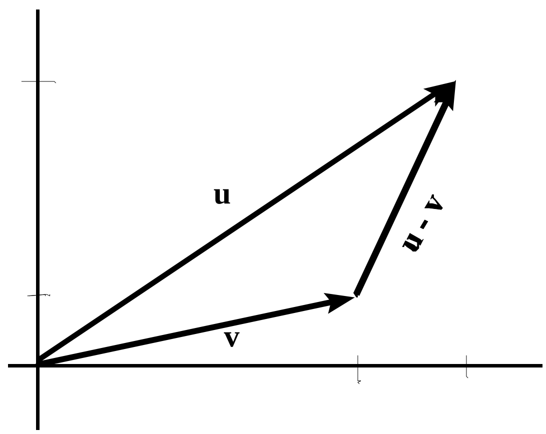
$$AI = A = IA$$

7. Given the vectors $\tilde{\mathbf{u}} = (a, b)$ and $\tilde{\mathbf{v}} = (c, d)$, give the geometric interpretation of the following:

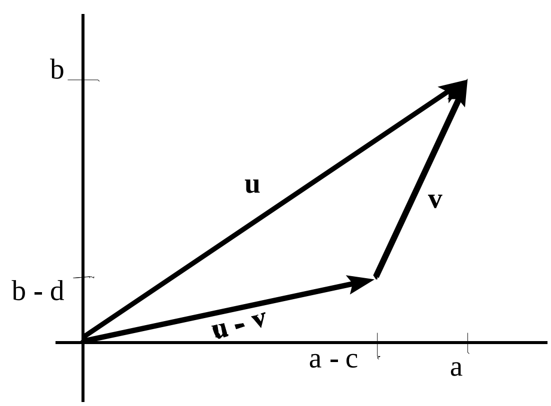
(a) $\tilde{\mathbf{u}} + \tilde{\mathbf{v}}$



(b) $\tilde{\mathbf{u}} - \tilde{\mathbf{v}}$



ALTERNATIVELY:



8. For

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & 6 & 3 \\ 2 & 1 & 0 \\ 9 & 8 & 5 \end{bmatrix}$$

compute $3A + 4B - 2C$.

$$\begin{aligned} 3A + 4B - 2C &= 3\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + 4\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 7 & 6 & 3 \\ 2 & 1 & 0 \\ 9 & 8 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 & 0 \\ 6 & 3 & 0 \\ 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} -4 & 4 & 4 \\ 4 & -4 & 4 \\ 8 & 12 & 16 \end{bmatrix} - \begin{bmatrix} 14 & 12 & 6 \\ 4 & 2 & 0 \\ 18 & 16 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -15 & -5 & -2 \\ 6 & -3 & 4 \\ -7 & -1 & 9 \end{bmatrix} \end{aligned}$$

9. For $n \times n$ matrices A, B, C, write the transpose of each matrix

Before doing these exercises, we should review three facts about matrix transposes:

$$\left(A^T\right)^T = A$$

$(A+B)^T = A^T + B^T$ (i.e., the transpose of a sum equals the sum of the transposes)

$(AB)^T = B^T A^T$ (i.e., the transpose of a product equals the product of the transposes – **in reverse order.)**

(a) A^T

$$\left(A^T\right)^T = A$$

(b) $A^T A$

$$\left(A^T A\right)^T = A^T \left(A^T\right)^T = A^T A$$

(c) $A(B+C)$

$$[A(B+C)]^T = (B+C)^T A^T = \left(B^T + C^T\right) A^T = B^T A^T + C^T A^T$$

(d) ABC^T

$$\left[ABC^T\right]^T = \left(C^T\right)^T (B)^T (A)^T = CB^T A^T$$

(e) $AB^T A^T$

$$\left[AB^T A^T\right]^T = \left(A^T\right)^T \left(B^T\right)^T (A)^T = ABA^T$$

(f) $A^T B + B^T A =$

$$\left[A^T B + B^T A\right]^T = \left[A^T B\right]^T + \left[B^T A\right]^T = (B)^T \left(A^T\right)^T + (A)^T \left(B^T\right)^T = B^T A + A^T B$$

10. For the matrices given below, calculate each of the following products when they are defined.

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) AB

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (2\cdot 1 + 1\cdot 0 + 2\cdot 1) & (2\cdot 1 + 1\cdot 1 + 2\cdot(-2)) & (2\cdot 0 + 1\cdot 1 + 2\cdot 1) \\ (3\cdot 1 + 1\cdot 0 + 2\cdot 1) & (3\cdot 1 + 1\cdot 1 + 2\cdot(-2)) & (3\cdot 0 + 1\cdot 1 + 2\cdot 1) \\ (1\cdot 1 + 0\cdot 0 + 1\cdot 1) & (1\cdot 1 + 0\cdot 1 + 1\cdot(-2)) & (1\cdot 0 + 0\cdot 1 + 1\cdot 1) \end{bmatrix} \\ &= \begin{bmatrix} 4 & -1 & 3 \\ 5 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \end{aligned}$$

(b) AC

$$\begin{aligned} AC &= \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} (2\cdot 2 + 1\cdot 1 + 2\cdot 4) & (2\cdot 1 + 1\cdot 1 + 2\cdot 0) \\ (3\cdot 2 + 1\cdot 1 + 2\cdot 4) & (3\cdot 1 + 1\cdot 1 + 2\cdot 0) \\ (1\cdot 2 + 0\cdot 1 + 1\cdot 4) & (1\cdot 1 + 0\cdot 1 + 1\cdot 0) \end{bmatrix} \\ &= \begin{bmatrix} 13 & 3 \\ 15 & 4 \\ 6 & 1 \end{bmatrix} \end{aligned}$$

(c) CD

$$CD = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (2\cdot 3 + 1\cdot 1) & (2\cdot 1 + 1\cdot 2) \\ (1\cdot 3 + 1\cdot 1) & (1\cdot 1 + 1\cdot 2) \\ (4\cdot 3 + 0\cdot 1) & (4\cdot 1 + 0\cdot 2) \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 4 & 3 \\ 12 & 4 \end{bmatrix}$$

(d) BC

$$BC = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1\cdot 2 + 1\cdot 1 + 0\cdot 4) & (1\cdot 1 + 1\cdot 1 + 0\cdot 0) \\ (0\cdot 2 + 1\cdot 1 + 1\cdot 4) & (0\cdot 1 + 1\cdot 1 + 1\cdot 0) \\ (1\cdot 2 + (-2)\cdot 1 + 1\cdot 4) & (1\cdot 1 + (-2)\cdot 1 + 1\cdot 0) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 4 & -1 \end{bmatrix}$$

(e) DC

$$DC = \underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 0 \end{bmatrix}}_{3\times 2} \text{ this is undefined } \quad 2 \times \underbrace{2 \text{ and } 3}_{\text{not the same}} \times 2$$

(f) $BC + CD$

$$BC + CD = \begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ 4 & 3 \\ 12 & 4 \end{bmatrix} = \begin{bmatrix} (3+7) & (2+4) \\ (5+4) & (1+3) \\ (4+12) & (-1+4) \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 9 & 4 \\ 16 & 3 \end{bmatrix}$$

(g) $AC + BC$

$$AC + BC = \begin{bmatrix} 13 & 3 \\ 15 & 4 \\ 6 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 5 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} (13+3) & (3+2) \\ (15+5) & (4+1) \\ (6+4) & (1-1) \end{bmatrix} = \begin{bmatrix} 16 & 5 \\ 20 & 5 \\ 10 & 0 \end{bmatrix}$$

(h) $(A+B)\,C$

$$(A+B)\,C = AC + BC = \begin{bmatrix} 16 & 5 \\ 20 & 5 \\ 10 & 0 \end{bmatrix} \quad \text{(From part 10.g)}$$

11. Given that $\tilde{\mathbf{u}} = (1,2,3,4)$ and $\tilde{\mathbf{v}} = (3,1,2,1)$, compute the dot product $\mathbf{u} \circ \mathbf{v}$.

$\mathbf{u} \circ \mathbf{v} = (1,2,3,4) \circ (3,1,2,1) = (3+2+6+4) = 15$