MTH 3331 - Test #2 - Solutions

Spring 2018

Pat Rossi Name ______________________

Show CLEARLY how you arrive at your answers!

1. Solve the system of equations:

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 13 \\ -x & + & y & - & 7z & = & -34 \\ 2x & + & 5y & - & 4z & = & -1 \end{array}$$

The system can be represented in augmented matrix form as:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 13 \\ -1 & 1 & -7 & -34 \\ 2 & 5 & -4 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 1 & 13 \\ 0 & 1 & -2 & -7 \\ 0 & 1 & -6 & -27 \end{array}\right]$$

Replace Row 2 with the sum of Row 2 + Row 1
Replace Row 3 with the sum of Row 3 + (-2)Row 1

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 1 & 13 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

Replace Row 3 with the sum of Row 3 + (-1)Row 2

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 1 & 13 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Multiply Row 3 by $\left(-\frac{1}{4}\right)$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Replace Row 1 with the sum of Row 1 + (-1)Row 3
Replace Row 2 with the sum of Row 2 + (2)Row 3

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Replace Row 1 with the sum of Row 1 + (-2)Row 2

This yields the **solution:** $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

2. Give the general solution to the system of equations:

$$\begin{array}{rcrcrcrcr} 2x_1 & & & + & 4x_3 & + & 16x_4 & = & -2 \\ -2x_1 & + & x_2 & - & 6x_3 & - & 22x_4 & = & 4 \\ 4x_1 & - & 2x_2 & + & 10x_3 & + & 40x_4 & = & -10 \end{array}$$

The system can be represented in augmented matrix form as:

$$\underbrace{\left[\begin{array}{rrrr} 2 & 0 & 4 & 16 \\ -2 & 1 & -6 & -22 \\ 4 & -2 & 10 & 40 \end{array}\right.}_{[A]}\underbrace{\left|\begin{array}{r} -2 \\ 4 \\ -10 \end{array}\right]}_{[\vec{b}]}$$

$$\Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 8 & -1 \\ -2 & 1 & -6 & -22 & 4 \\ 4 & -2 & 10 & 40 & -10 \end{array}\right]$$ Multiply Row 1 by $\frac{1}{2}$

$$\Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 8 & -1 \\ 0 & 1 & -2 & -6 & 2 \\ 0 & -2 & 2 & 8 & -6 \end{array}\right]$$ Replace Row 2 with the sum of Row 2 + (2)Row 1
Replace Row 3 with the sum of Row 3 + (−4)Row 1

$$\Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 8 & -1 \\ 0 & 1 & -2 & -6 & 2 \\ 0 & 0 & -2 & -4 & -2 \end{array}\right]$$ Replace Row 3 with the sum of Row 3 + (2)Row 2

$$\Rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 8 & -1 \\ & 1 & -2 & -6 & 2 \\ & & 1 & 2 & 1 \end{array}\right]$$ Multiply Row 3 by $-\frac{1}{2}$

$$\Rightarrow \underbrace{\left[\begin{array}{rrrr} 1 & 0 & 0 & 4 \\ & 1 & 0 & -2 \\ & & 1 & 2 \end{array}\right.}_{[A^*]}\underbrace{\left|\begin{array}{r} -3 \\ 4 \\ 1 \end{array}\right]}_{[\vec{b}^*]}$$ Replace Row 1 with the sum of Row 1 + (−2)Row 3
Replace Row 2 with the sum of Row 2 + (2)Row 3

We identify the pivot variables and free variables.

Pivot	**Free**
x_1	
x_2	x_4
x_3	

To find a particular solution of the system, set the free variables equal to zero.

Thus, Equation 1 becomes: $x_1 + 4\,(0) = -3$ i.e., $x_1 = -3$

Equation 2 becomes: $x_2 - 2\,(0) = 4$ i.e., $x_2 = 4$

Equation 3 becomes: $x_3 + 2\,(0) = 1$ i.e., $x_3 = 1$

Our particular solution is given by:

$$\vec{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

It now remains for us to find the homogeneous solution.

This is the solution to the related system of equations $[A]\,[\vec{x}] = \left[\vec{0}\right]$

$$\text{Given the system:}\quad \begin{array}{rcrcrcrcl} 2x_1 & & & + & 4x_3 & + & 16x_4 & = & 0 \\ -2x_1 & + & x_2 & - & 6x_3 & - & 22x_4 & = & 0 \\ 4x_1 & - & 2x_2 & + & 10x_3 & + & 40x_4 & = & 0 \end{array}$$

$$\left[\begin{array}{rrrr|r} 2 & 0 & 4 & 16 & 0 \\ -2 & 1 & -6 & -22 & 0 \\ 4 & -2 & 10 & 40 & 0 \end{array}\right], \text{ which reduces to } \underbrace{\left[\begin{array}{rrrr} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right.}_{[A^*]}\underbrace{\left|\begin{array}{r} 0 \\ 0 \\ 0 \end{array}\right]}_{[\vec{0}]},$$

using the exact same sequence of row operations that was used to find the particular solution.

We parameterize the free variables.

$\Rightarrow x_4 = k$

Thus, Equation 1 becomes: $x_1 + 4k = 0$ i.e., $x_1 = -4k$

Equation 2 becomes: $x_2 - 2k = 0$ i.e., $x_2 = 2k$

Equation 3 becomes: $x_3 + 2k = 0$ i.e., $x_3 = -2k$

Our homogeneous solution is given by:

$$\vec{x}_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4k \\ 2k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} -4 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

Our general solution is given by $\vec{x} = \vec{x}_p + \vec{x}_h = \begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -4 \\ 2 \\ -2 \\ 1 \end{bmatrix}$

3. A system of equations has been reduced to "row echelon form (below)." Give the general solution of the system, if the variables are x_1, x_2, x_3, x_4.

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right]}_{[A^*] \quad [\vec{b}^*]}$$

We identify the pivot variables and free variables.

Pivot	**Free**
x_1	
x_2	x_3
x_4	

To find a particular solution of the system, set the free variables equal to zero.

Thus, Equation 1 becomes: $x_1 + 2(0) = 5$ i.e., $x_1 = 5$

Equation 2 becomes: $x_2 - 2(0) = -3$ i.e., $x_2 = -3$

Equation 3 becomes: $x_4 = 8$ i.e., $x_4 = 8$

Our particular solution is given by:

$$\vec{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 8 \end{bmatrix}$$

It now remains for us to find the homogeneous solution.

This is the solution to the related system of equations $[A][\vec{x}] = [\vec{0}]$, which reduces to:

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]}_{[A^*] \quad [\vec{b}^*]}.$$

We parameterize the free variables.

$\Rightarrow x_3 = k$

Thus, Equation 1 becomes: $x_1 + 2k = 0$ i.e., $x_1 = -2k$

Equation 2 becomes: $x_2 - 2k = 0$ i.e., $x_2 = 2k$

Equation 3 becomes: $x_4 = 0$ i.e., $x_4 = 0$

Our homogeneous solution is given by:

$$\vec{x}_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2k \\ 2k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Our general solution is given by $\vec{x} = \vec{x}_p + \vec{x}_h = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 8 \end{bmatrix} + k \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

4. ~

(a) If $\vec{x}_1$ is a solution of the system of equations $[A]\,[\vec{x}] = \left[\vec{b}\right]$, and $\vec{y}_1$ is a solutions of the corresponding homogeneous system $[A]\,[\vec{x}] = \left[\vec{0}\right]$, what can we say about $(\vec{x}_1 + \vec{y}_1)$, and why?

Answer: $(\vec{x}_1 + \vec{y}_1)$ is also a solution of the system of equations $[A]\,[\vec{x}] = \left[\vec{b}\right]$.

Here's why:

$[A]\,[(\vec{x}_1 + \vec{y}_1)] = [A]\,[\vec{x}_1] + [A]\,[\vec{y}_1] = \left[\vec{b}\right] + \left[\vec{0}\right] = \left[\vec{b}\right]$

i.e., $[A]\,[(\vec{x}_1 + \vec{y}_1)] = \left[\vec{b}\right]$

(b) If $\vec{x}_1$ and $\vec{x}_2$ are both solutions of the system of equations $[A]\,[\vec{x}] = \left[\vec{b}\right]$, what can we say about $(\vec{x}_1 - \vec{x}_2)$, and why?

Answer: $(\vec{x}_1 - \vec{x}_2)$ is a solution of the corresponding homogeneous system $[A]\,[\vec{x}] = \left[\vec{0}\right]$.

Here's why:

$[A]\,[(\vec{x}_1 - \vec{x}_2)] = [A]\,[\vec{x}_1] - [A]\,[\vec{x}_2] = \left[\vec{b}\right] - \left[\vec{b}\right] = \left[\vec{0}\right]$

i.e., $[A]\,[(\vec{x}_1 - \vec{x}_2)] = \left[\vec{0}\right]$

(c) If $\vec{y}_1$ and $\vec{y}_2$ are both solutions of the system of equations $[A]\,[\vec{x}] = \left[\vec{0}\right]$, what can we say about $(\vec{y}_1 + \vec{y}_2)$, and why?

Answer: $(\vec{y}_1 + \vec{y}_2)$ is also a solution of the system of equations $[A]\,[\vec{x}] = \left[\vec{0}\right]$.

Here's why:

$[A]\,[(\vec{y}_1 + \vec{y}_2)] = [A]\,[\vec{y}_1] - [A]\,[\vec{y}_2] = \left[\vec{0}\right] + \left[\vec{0}\right] = \left[\vec{0}\right]$

i.e., $[A]\,[(\vec{y}_1 + \vec{y}_2)] = \left[\vec{0}\right]$