MTH 3331 Test #2 - Solutions

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Instructions. Show clearly how you arrive at your answers.

1. State the Cauchy-Schwarz Inequality.

   For vectors, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ in $\mathbf{R}^n$,

   $$|\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}| \leq ||\tilde{\mathbf{u}}|| \, ||\tilde{\mathbf{v}}||$$

   with equality exactly when $\tilde{\mathbf{u}} = k\tilde{\mathbf{v}}$, for any scalar, k.

2. State the Triangle Inequality and give its geometric interpretation.

   For vectors, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ in $\mathbf{R}^n$,

   $$||\tilde{\mathbf{u}} + \tilde{\mathbf{v}}|| \leq ||\tilde{\mathbf{u}}|| + ||\tilde{\mathbf{v}}||$$

   with equality exactly when $\tilde{\mathbf{u}} = k\tilde{\mathbf{v}}$, for any scalar, $k > 0$.

   This means that no side of a triangle has greater length that the sums of the lengths of the other two sides.

3. Find the norm or "length" of the vectors $\tilde{\mathbf{u}} = (3, 4, 12)$ and $\tilde{\mathbf{v}} = (4, 4, 5, -2)$

   $||\tilde{\mathbf{u}}|| = ||(3, 4, 12)|| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$

   $||\tilde{\mathbf{v}}|| = ||(4, 4, 5, -2)|| = \sqrt{4^2 + 4^2 + 5^2 + (-2)^2} = \sqrt{61}$

4. Given that $\tilde{\mathbf{u}} = (1, 2, 3, 4)$ and $\tilde{\mathbf{v}} = (3, 1, 2, 1)$, compute the dot product $\mathbf{u} \circ \mathbf{v}$.

   $\mathbf{u} \circ \mathbf{v} = (1, 2, 3, 4) \circ (3, 1, 2, 1) = (3 + 2 + 6 + 4) = 15$

5. Find $\cos(\theta)$, where θ is the angle between vectors $\tilde{\mathbf{u}} = (2, 4, 1)$ and $\tilde{\mathbf{v}} = (-2, 2, -4)$

   $\cos(\theta) = \frac{\mathbf{u} \circ \mathbf{v}}{||\tilde{\mathbf{u}}|| \, ||\tilde{\mathbf{v}}||} = \frac{(2,4,1) \circ (-2,2,-4)}{||(2,4,1)|| \, ||(-2,2,-4)||} = \frac{0}{\sqrt{21}\sqrt{24}} = 0$

   i.e., $\cos(\theta) = 0$

6. Find a unit vector that has the same direction as the vector $(-1,4,3)$.

   (a) To form such a vector, we divide this vector by its length.

   $$\tilde{\mathbf{u}} = \frac{(-1,4,3)}{||(-1,4,3)||} = \frac{(-1,4,3)}{\sqrt{1+16+9}} = \left(-\frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{3}{\sqrt{26}}\right) = \left(\frac{\sqrt{26}}{26}, \frac{2\sqrt{26}}{13}, \frac{3\sqrt{26}}{26}\right)$$

7. Let $\mathbf{u} = (1,1,2)$ and $\mathbf{v} = (k,0,3)$. Choose k such that: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

   (a) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

   If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\mathbf{u} \circ \mathbf{v} = 0$

   So we want, $\mathbf{u} \circ \mathbf{v} = (1,1,2) \circ (k,0,3) = k + 6 = 0$

   $\Rightarrow k = -6$

8. Solve the system of equations below:

$$\begin{bmatrix} x_1 & + & x_2 & + & 2x_3 & = & 9 \\ 2x_1 & + & 4x_2 & - & 3x_3 & = & 1 \\ 3x_1 & + & 6x_2 & - & 5x_3 & = & 0 \end{bmatrix}$$

Form the augmented matrix, and reduce:

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

9. Compute A^{-1} if

$$A = \begin{bmatrix} 3 & 2 & -1 \\ -2 & 6 & \frac{7}{6} \\ -1 & -1 & \frac{1}{3} \end{bmatrix}$$

Form the augmented matrix, and reduce:

$$\left[\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ -2 & 6 & \frac{7}{6} & 0 & 1 & 0 \\ -1 & -1 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ -2 & 6 & \frac{7}{6} & 0 & 1 & 0 \\ -1 & -1 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{22}{3} & \frac{1}{2} & \frac{2}{3} & 1 & 0 \\ -1 & -1 & \frac{1}{3} & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{22}{3} & \frac{1}{2} & \frac{2}{3} & 1 & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{3}{44} & \frac{1}{11} & \frac{3}{22} & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{3}{44} & \frac{1}{11} & \frac{3}{22} & 0 \\ 0 & & \frac{1}{44} & \frac{4}{11} & \frac{1}{22} & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{3}{44} & \frac{1}{11} & \frac{3}{22} & 0 \\ 0 & & 1 & 16 & 2 & 44 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -3 \\ 0 & & 1 & 16 & 2 & 44 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & 0 & \frac{17}{3} & \frac{2}{3} & \frac{44}{3} \\ 0 & 1 & 0 & -1 & 0 & -3 \\ 0 & & 1 & 16 & 2 & 44 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{19}{3} & \frac{2}{3} & \frac{50}{3} \\ 0 & 1 & 0 & -1 & 0 & -3 \\ 0 & & 1 & 16 & 2 & 44 \end{bmatrix}$$

i.e., $A^{-1} = \begin{bmatrix} \frac{19}{3} & \frac{2}{3} & \frac{50}{3} \\ -1 & 0 & -3 \\ 16 & 2 & 44 \end{bmatrix}$

10. Solve the systems below, in the most economical way possible:

Since the coefficient ,matrix in each system is the matrix from the previous problem, the most convenient way to solve the system is to use A^{-1} from the previous problem, as follows: $[A]\,[\vec{x}] = \left[\vec{b}\right] \Rightarrow [\vec{x}] = [A^{-1}]\left[\vec{b}\right]$

(a)

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 20 \\ -2x_1 & + & 6x_2 & + & \frac{7}{6}x_3 & = & 5 \\ -x_1 & - & x_2 & + & \frac{1}{3}x_3 & = & 5 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{19}{3} & \frac{2}{3} & \frac{50}{3} \\ -1 & 0 & -3 \\ 16 & 2 & 44 \end{bmatrix} \begin{bmatrix} 20 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{640}{3} \\ -35 \\ 550 \end{bmatrix}$$

(b)

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 10 \\ -2x_1 & + & 6x_2 & + & \frac{7}{6}x_3 & = & 10 \\ -x_1 & - & x_2 & + & \frac{1}{3}x_3 & = & 10 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{19}{3} & \frac{2}{3} & \frac{50}{3} \\ -1 & 0 & -3 \\ 16 & 2 & 44 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{710}{3} \\ -40 \\ 620 \end{bmatrix}$$

(c)

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & -2 \\ -2x_1 & + & 6x_2 & + & \frac{7}{6}x_3 & = & 2 \\ -x_1 & - & x_2 & + & \frac{1}{3}x_3 & = & 6 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{19}{3} & \frac{2}{3} & \frac{50}{3} \\ -1 & 0 & -3 \\ 16 & 2 & 44 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{266}{3} \\ -16 \\ 236 \end{bmatrix}$$

11. Determine the equation of the plane which contains the point $(3, -1, 2)$, and which has normal vector $\langle 1, -1, 5 \rangle$.

First, note that if $\langle x, y, z \rangle$ is any point in the plane, then $\langle x - 3, y - (-1), z - 2 \rangle = \langle x - 3, y + 1, z - 2 \rangle$ is a vector in the plane. Hence, vectors $\langle 1, -1, 5 \rangle$ and $\langle x - 3, y + 1, z - 2 \rangle$ must be perpendicular. Thus, $\langle 1, -1, 5 \rangle \circ \langle x - 3, y + 1, z - 2 \rangle = 0$.

$\Rightarrow (x - 3) - (y + 1) + 5(z - 2) = 0$

$\Rightarrow x - 3 - y - 1 + 5z - 10 = 0$

$\Rightarrow x - y + 5z = 14$ is the equation of the plane that we seek.

12. Find the parametric equations of the line which contains the points $(3, 1, -2)$ and $(1, -1, 5)$.

Note that if $(3, 1, -2)$ and $(1, -1, 5)$ are points on the line that we seek,

then $\langle 3 - 1, 1 - (-1), -2 - 5 \rangle = \langle 2, 2, -7 \rangle$ is a vector that is parallel to the line that we seek.

Also, if (x, y, z) is any point on the line that we seek, then $\langle x - 1, y - (-1), z - 5 \rangle = \langle x - 1, y + 1, z - 5 \rangle$ is a vector parallel to that line.

Hence, $\langle x - 1, y + 1, z - 5 \rangle = k \langle 2, 2, -7 \rangle$ for some scalar, k.

$\Rightarrow \langle x - 1, y + 1, z - 5 \rangle = \langle 2k, 2k, -7k \rangle$

$$\Rightarrow \begin{array}{rcr} x - 1 & = & 2k \\ y + 1 & = & 2k \\ z - 5 & = & -7k \end{array}$$

$$\Rightarrow \begin{array}{rcr} x & = & 2k + 1 \\ y & = & 2k - 1 \\ z & = & -7k + 5 \end{array}$$ are the parametric equations of the line that we seek.

13. Solve the system of equations by reducing the augmented matrix to row-reduced echelon form.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & 2 \\ x_1 & - & x_2 & + & 4x_3 & + & 4x_4 & = & -4 \\ 2x_1 & + & x_2 & + & 5x_3 & + & 5x_4 & = & -2 \\ 3x_1 & & & + & 9x_3 & + & 9x_4 & = & -6 \end{array}$$

Form the augmented matrix, and reduce:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 1 & -1 & 4 & 4 & -4 \\ 2 & 1 & 5 & 5 & -2 \\ 3 & 0 & 9 & 9 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & -3 & 3 & 3 & -6 \\ 0 & -3 & 3 & 3 & -6 \\ 0 & -6 & 6 & 6 & -12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & -3 & 3 & 3 & -6 \\ 0 & -6 & 6 & 6 & -12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 3 & -2 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Variables	**Free Variables**
x_1	x_3
x_2	x_4

Find a particular solution by setting the free variables equal to zero.

Eq. 1 yields $x_1 = -2$

Eq. 2 yields $x_2 = 2$

$$\Rightarrow \mathbf{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{is a particular solution}$$

Next, form the related homogeneous system.

$$\begin{bmatrix} 1 & 0 & 3 & 3 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To parameterize the free variables.

Let $x_3 = k_1$

and $x_4 = k_2$

Eq. 1 yields:

$x_1 + 3x_3 + 3x_4 = 0$

$\Rightarrow x_1 = -3x_3 - 3x_4$

$\Rightarrow x_1 = -3k_1 - 3k_2$

Eq. 2 yields:

$x_2 - x_3 - x_4 = 0$

$\Rightarrow x_2 = x_3 + x_4$

$\Rightarrow x_2 = k_1 + k_2$

Our general homogeneous solutions is

$$x_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = k_1 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Our general solution is given by:

$$\mathbf{x}_g = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{x}_p + \mathbf{x}_h = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

14. Given the system

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

the row-reduced form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Write the solution set of this system as a linear combination of 3-tuples.

Pivot Variables	**Free Variables**
x_1	x_3
x_2	

Find a particular solution by setting the free variables equal to zero.

Eq. 1 yields $x_1 = 5$

Eq. 2 yields $x_2 = -1$

$$\Rightarrow \mathbf{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} \quad \text{is a particular solution}$$

Next, form the related homogeneous system.

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

To parameterize the free variables.

Let $x_3 = k_1$

Eq. 1 yields:

$x_1 - 2x_3 = 0$

$\Rightarrow x_1 = 2x_3$

$\Rightarrow x_1 = 2k_1$

Eq. 2 yields:

$x_2 + x_3 = 0$

$\Rightarrow x_2 = -x_3$

$\Rightarrow x_2 = -k_1$

Our general homogeneous solutions is

$$x_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Our general solution is given by:

$$\mathbf{x}_g = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}_p + \mathbf{x}_h = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$