MTH 3331 Test #3 - Solutions

Spring 2018

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Show CLEARLY how you arrive at your answers!

1. Verify Cauchy-Schwarz Inequality with the vectors $\tilde{\mathbf{u}} = \langle 1, 2, 2, 4 \rangle$; $\tilde{\mathbf{v}} = \langle 1, 1, 1, 1 \rangle$

   The Cauchy-Schwarz Inequality states that: Given vectors $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ in $\mathbb{R}^n$, $|\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}| \leq ||\tilde{\mathbf{u}}|| \, ||\tilde{\mathbf{v}}||$, with equality holding exactly when $\tilde{\mathbf{u}} = k\tilde{\mathbf{v}}$, where k is a scalar.

   $|\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}| = |\langle 1, 2, 2, 4 \rangle \circ \langle 1, 1, 1, 1 \rangle| = |1 + 2 + 2 + 4| = 9$

   $||\tilde{\mathbf{u}}|| = \sqrt{1^2 + 2^2 + 2^2 + 4^2} = \sqrt{25} = 5$

   $||\tilde{\mathbf{v}}|| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$

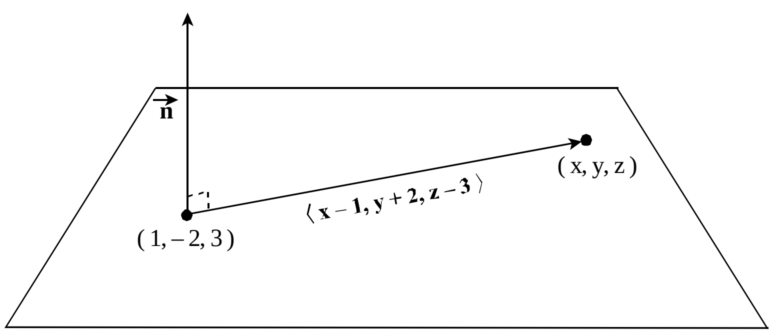
   Observe: $|\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}| = 9 \leq 10 = 5 \cdot 2 = ||\tilde{\mathbf{u}}|| \, ||\tilde{\mathbf{v}}||$

   i.e., $|\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}| \leq ||\tilde{\mathbf{u}}|| \, ||\tilde{\mathbf{v}}||$

2. Give the equation of the plane that contains the point $(1, -2, 3)$, and which has, as a normal vector $\tilde{\mathbf{n}} = \langle 2, 1, 1 \rangle$

   If $(1, -2, 3)$ is a point in the plane and (x, y, z) is any point in the plane, then $\langle x - 1, y - (-2), z - 3 \rangle = \langle x - 1, y + 2, z - 3 \rangle$ is a vector in the plane, going from the point $(1, -2, 3)$ to the point (x, y, z).

   Furthermore, since $\tilde{\mathbf{n}}$ is a vector normal (orthogonal) to the plane, it is orthogonal (perpendicular) to *every* vector in the plane, including the vector $\langle x - 1, y + 2, z - 3 \rangle$. (See below.)



   Since $\tilde{\mathbf{n}}$ is orthogonal to $\langle x - 1, y + 2, z - 3 \rangle$, it follows that $\tilde{\mathbf{n}} \circ \langle x - 1, y + 2, z - 3 \rangle = 0$.

   i.e., $\langle 2, 1, 1 \rangle \circ \langle x - 1, y + 2, z - 3 \rangle = 0$

   $\Rightarrow \langle 2, 1, 1 \rangle \circ \langle x - 1, y + 2, z - 3 \rangle = 2(x - 1) + 1(y + 2) + 1(z - 3) = 0$

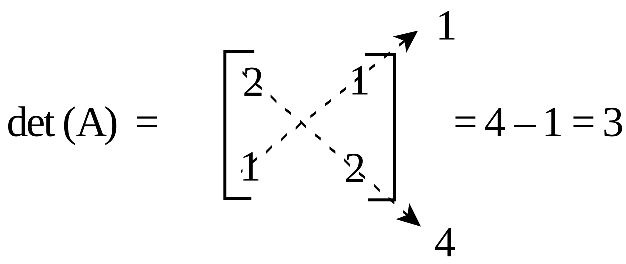
   $\Rightarrow 2x - 2 + y + 2 + z - 3 = 0$

   $\Rightarrow 2x + y + z = 3$

The equation of the plane is $2x + y + z = 3$

3. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; compute $\det(A)$

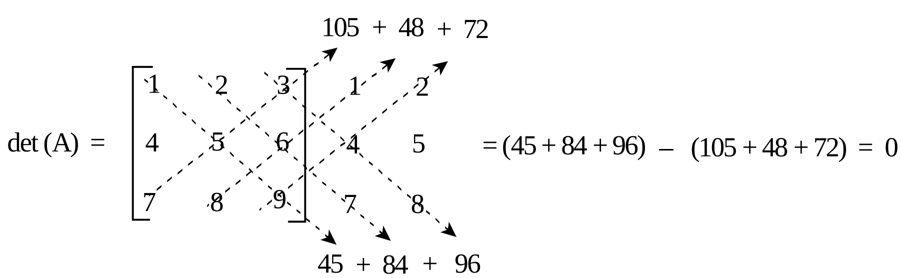
The computation of $\det(A)$ is show schematically below:



$\det(A) = 3$

4. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$; compute $\det(A)$

The computation of $\det(A)$ is show schematically below:



$\det(A) = 0$

5. $A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 4 & 0 & 0 & 0 \\ 4 & 3 & 2 & 3 \end{bmatrix}$; compute $\det(A)$

Since A is larger than 3×3, the only way to compute $\det(A)$ is by the "cofactor method."

Since row 3 has zeros, it will be most efficient if we compute $\det(A)$ by expanding along row 3.

$\det(A) = a_{31}(-1)^{3+1}\det(A_{31}) + a_{32}(-1)^{3+2}\det(A_{32}) + a_{33}(-1)^{3+3}\det(A_{33}) + a_{34}(-1)^{3+4}\det(A_{34})$

$= 4 \cdot (-1)^{3+1}\det(A_{31}) + 0 \cdot (-1)^{3+2}\det(A_{32}) + 0 \cdot (-1)^{3+3}\det(A_{33}) + 0 \cdot (-1)^{3+4}\det(A_{34})$

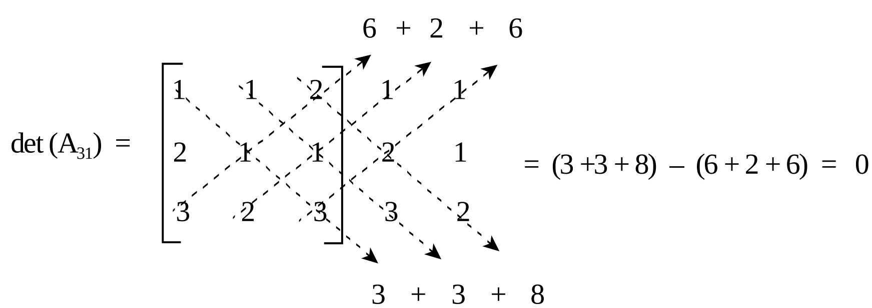
$= 4 \cdot (-1)^{3+1}\det(A_{31})$

To continue our computation, we must compute the determinant of the "minor" A_{31}.

The "minor" A_{31} is formed by taking the matrix A and removing row (3) and col (1).

$A_{31} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

Computation of $\det(A_{31})$ is shown schematically, below:



Continuing where we left off, we have:

$\det(A) = \ldots = 4 \cdot (-1)^{3+1}\det(A_{31}) = 4 \cdot 1 \cdot 0 = 0$

$\boxed{\det(A) = 0}$

Alternativley: We may have noticed that if we look closely at the rows of A, $\text{row}(1) + \text{row}(2) = \text{row}(4)$.

Hence, the rows of A are linearly dependent.

This means that $\det(A) = 0$.

6. Complete the list below, by adding at least 5 appropriate statements

   The following statements are equivalent for an $n \times n$ matrix A

   (a) A is nonsingular

   (b) A is row equivalent to I

   (c) A has rank n

   (d) A^{-1} exists

   (e) $\det(A) \neq 0$

   (f) The rows of A are linearly independent

   (g) The columns of A are linearly independent

   (h) The rows of A span $\mathbb{R}^n$

   (i) The columns of A span $\mathbb{R}^n$

   (j) The rows of A form a basis for $\mathbb{R}^n$

   (k) The columns of A form a basis for $\mathbb{R}^n$

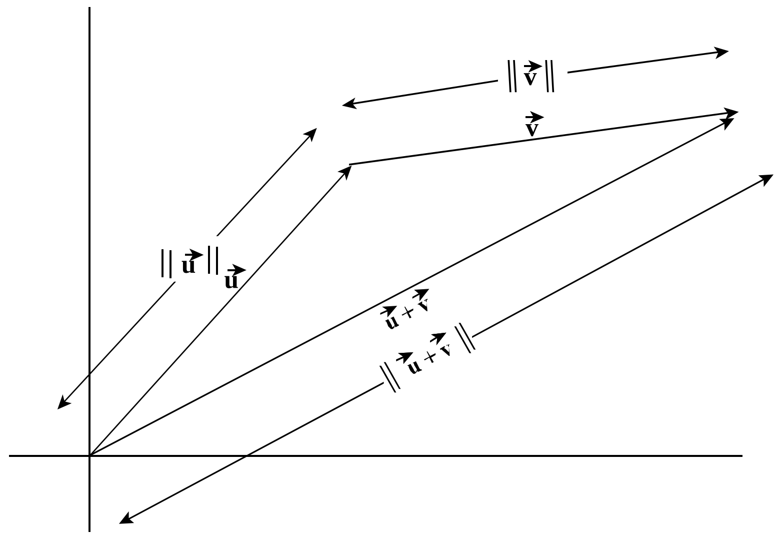
   (l) The system of equations $[\,A\,][\vec{x}] = \left[\vec{b}\right]$ has a unique solution.

   (m) The system of equations $[\,A\,][\vec{x}] = \left[\vec{0}\right]$ has only the trivial solution.

7. State the Triangle Iequality and give the geometric interpretation

   If $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ in $\mathbb{R}^n$, $||\tilde{\mathbf{u}} + \tilde{\mathbf{v}}|| \leq ||\tilde{\mathbf{u}}|| + ||\tilde{\mathbf{v}}||$, with equality holding exactly when $\tilde{\mathbf{u}} = k\tilde{\mathbf{v}}$, where k is a scalar.

   The goemetric interpretation of this statement is: No one side of a triangle $(\tilde{\mathbf{u}} + \tilde{\mathbf{v}})$ has length greater than the sum of the lengths of the other two sides $(||\tilde{\mathbf{u}}|| + ||\tilde{\mathbf{v}}||)$. (See the illustration below.)



8. Characterize the rows of each matrix as being singular or nonsingular. Justify your answer.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(nonsingular) The determinant is non-zero. $(det\,(A) = -2)$. Alternatively, one row is not a multiple of the other, which is the only way that a pair of rows can be linarly dependent.

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 5 & 1 & 6 \end{bmatrix}$

(singular) The columns are linearly dependent. $(\mathrm{col}\,(1) + \mathrm{col}\,(2) = \mathrm{col}\,(3))$

(c) $\begin{bmatrix} 3 & 8 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

(nonsingular) The matrix is clearly seen to be row equivalent to the identity matrix I

(d) $\begin{bmatrix} 8 & 2 & 1 & -1 \\ 4 & -3 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 2 \end{bmatrix}$

(singular) The rows of the matrix are dependent, since one of the rows is the zero row.

(e) $\begin{bmatrix} 3 & 0 & 3 \\ 0 & 2 & 0 \\ 2 & 0 & 5 \end{bmatrix}$

(nonsingular). $(\det\,(A) \neq 0.\ (\det\,(A) = 18))$