# MTH 3318 Solutions to Induction Problems - Set 1

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**Instructions.** Prove the following by mathematical induction.

**Set 1 (1):** $1+2+3+\ldots+n=\frac{n(n+1)}{2};\ \forall n\in\mathbf{N}$

i.e. $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$

**Proof.**

**Step #1:** Show that the proposition is true for $n=1$.

$\sum_{i=1}^{1} i=1=\frac{(1)((1)+1)}{2}$ True.

**Step #2:** Assume that the proposition is true for $n=k$, and prove that the proposition is true for $n=k+1$.

i.e., Assume that $\sum_{i=1}^{k} i=\frac{k(k+1)}{2}$ is true for some natural number $k$, and prove that $\sum_{i=1}^{k+1} i=\frac{(k+1)((k+1)+1)}{2}$ is true. (Equivalently, prove that $\sum_{i=1}^{k+1} i=\frac{k^2+3k+2}{2}$.)

**Observe:** $\sum_{i=1}^{k+1} i=\underbrace{\sum_{i=1}^{k} i+(k+1)=\frac{k(k+1)}{2}+(k+1)}_{\text{by Induction Hypothesis}}=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$

$=\frac{k^2+k}{2}+\frac{2k+2}{2}=\frac{k^2+3k+2}{2}$.

i.e., $\sum_{i=1}^{k+1} i=\frac{k^2+3k+2}{2}$.

Hence, $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$ for all natural numbers, $n$. ■

**Set 1 (2):** $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$; $\forall n \in \mathbf{N}$

i.e. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

**Proof.**

**Step #1:** Show that the proposition is true for $n = 1$.

$\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{(1)[(1)+1][2(1)+1]}{6}$. True.

**Step #2:** Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

i.e., Assume that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ and show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$.

i.e., show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

**Observe:**

$$\sum_{i=1}^{k+1} i^2 = \underbrace{\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2}_{\text{by Induction Hypothesis}} = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)\left[2k^2+7k+6\right]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

i.e., $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Hence, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$; $\forall n \in \mathbf{N}$ ■

**Set 1 (3):** $1 + x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1}-1}{x-1}$; $\forall n \in \mathbf{N} \cup \{0\}$; where $x \neq 1$.

i.e. $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$; where $x \neq 1$.

**Proof.**

**Step #1:** Show that the proposition is true for $n = 0$.

$\sum_{i=0}^{0} x^i = x^{(0)} = 1 = \frac{x-1}{x-1} = \frac{x^{(0)+1}-1}{x-1}$. True.

**Step #2:** Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

i.e., Assume that $\sum_{i=0}^{k} x^i = \frac{x^{k+1}-1}{x-1}$ and show that
$\sum_{i=0}^{k+1} x^i = \frac{x^{(k+1)+1}-1}{x-1}$.

i.e., show that $\sum_{i=0}^{k+1} x^i = \frac{x^{k+2}-1}{x-1}$

**Observe:**

$$\sum_{i=0}^{k+1} x^i = \underbrace{\sum_{i=0}^{k} x^i + x^{k+1} = \frac{x^{k+1}-1}{x-1} + x^{k+1}}_{\text{by Induction Hypothesis}} = \frac{x^{k+1}-1}{x-1} + \frac{x^{k+1}(x-1)}{x-1}$$

$$= \frac{x^{k+1}-1}{x-1} + \frac{x^{k+2}-x^{k+1}}{x-1} = \frac{x^{k+1}-1+x^{k+2}-x^{k+1}}{x-1} = \frac{x^{k+2}-1}{x-1}$$

i.e., $\sum_{i=0}^{k+1} x^i = \frac{x^{k+2}-1}{x-1}$

Hence, $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$; $\forall n \in \mathbf{N} \cup \{0\}$ ■

Remark: In this example, we allowed 0 to be an " honorary member" of the natural numbers. If we want to prove a proposition $P(n)$ $\forall n \in \mathbf{N} \cup \{0\}$, we prove $P(n)$ true for $n = 0$ on the first induction step.

**Set 1 (4):** Given that $|x_1 + x_2| \leq |x_1| + |x_2|$ (the Triangle Inequality); Prove by induction that:

$|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ (the General Triangle Inequality).

**Proof.**

**Step #1:** Show that the proposition is true for $n = 1$.

$|x_1| \leq |x_1|$.   True.

**Step #2:** Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

i.e., Assume that $|x_1 + x_2 + x_3 + \ldots + x_k| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k|$ and show that $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

**Observe:** $\underbrace{|(x_1 + x_2 + x_3 + \ldots + x_k) + x_{k+1}| \leq |x_1 + x_2 + x_3 + \ldots + x_k| + |x_{k+1}|}_{\text{from given}}$

$\leq \underbrace{|x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|}_{\text{by Ind. Hyp.}}$.

i.e., $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

Hence, $|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ for all natural numbers, $n$. ■

**Set 1 (5):** $(1+x)^n \geq 1+nx$ for any natural number $n$ and any real number $x \geq -1$.

**Proof.**

**Step #1:** Show true for $n=1$

$(1+x)^1 = 1+x \geq 1+(1)\,x$ True.

**Step #2:** Assume true for $n=k$, and show true for $n=k+1$

i.e., Assume that $(1+x)^k \geq 1+kx$ for some natural number $k$, and show that

$(1+x)^{k+1} \geq 1+(k+1)\,x$

**Observe:**

$$(1+x)^{k+1} = \underbrace{(1+x)^k(1+x) \geq (1+kx)(1+x)}_{\text{by Induction Hypothesis}} = 1+kx+x+kx^2$$

$$= 1+(k+1)\,x + \underbrace{kx^2}_{kx^2 \geq 0} \geq 1+(k+1)\,x$$

i.e., $(1+x)^{k+1} \geq 1+(k+1)\,x$

Hence, $(1+x)^n \geq 1+nx$ for all natural numbers $n$ and any real number $x \geq -1$ ■

Remark: Our proof hinged on two subtle points:

First, since $k$ is a natural number (hence greater than zero) and $x^2 \geq 0$ for ALL real numbers $x$, it follows that $kx^2 \geq 0$.

Second, since it is given that $x \geq -1$ (or equivalently, $(1+x) \geq 0$), the direction of the inequality, $(1+x)^k \geq 1+kx$, is preserved when both sides are multiplied by $(1+x)$ during the application of the induction hypothesis.

**Set 1 (6):** For $0 \le a \le b$; prove that $a^n \le b^n$.

**Proof.**

**Step #1:** Show true for $n = 1$.

$$a^1 = \underbrace{a \le b}_{\text{given}} = b^1 \qquad \text{True.}$$

i.e., $a^1 \le b^1$

**Step #2:** Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $a^k \le b^k$ for some natural number $k$, and show that $a^{k+1} \le b^{k+1}$

**Observe:** $a^{k+1} = a^k \cdot a \le \underbrace{b^k \cdot a}_{\text{by Ind. Hyp.}} \le \underbrace{b^k \cdot b}_{a \le b} = b^{k+1}$

i.e., $a^{k+1} \le b^{k+1}$

Hence, $a^n \le b^n$ for all natural numbers, $n$. ■

**Set 1 (7):** $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

i.e. $\sum_{i=1}^{n} (2i - 1) = n^2$

**Proof.**

**Step #1:** Show true for $n = 1$

$\sum_{i=1}^{1} (2i - 1) = (2(1) - 1) = 1 = (1)^2$ True.

**Step #2:** Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\sum_{i=1}^{k} (2i - 1) = k^2$ for some natural number $k$, and show that $\sum_{i=1}^{k+1} (2i - 1) = (k + 1)^2$

**Observe:**

$$\sum_{i=1}^{k+1} (2i - 1) = \underbrace{\sum_{i=1}^{k} (2i - 1) + (2(k + 1) - 1) = k^2 + (2(k + 1) - 1)}_{\text{by Induction Hypothesis}}$$

$$= k^2 + 2k + 1 = (k + 1)^2$$

i.e., $\sum_{i=1}^{k+1} (2i - 1) = (k + 1)^2$

Hence, $\sum_{i=1}^{n} (2i - 1) = n^2$ for all natural numbers, $n$. ■

**Set 1 (8):** $2+4+6+\ldots+2n=n^2+n;\ \forall n\in\mathbf{N}$

i.e. $\sum_{i=1}^{n}2i=n^2+n$

**Proof.**

**Step #1:** Show true for $n=1$

$\sum_{i=1}^{1}2i=2\left(1\right)=2=\left(1\right)^2+\left(1\right)$ True.

**Step #2:** Assume true for $n=k,$ and show true for $n=k+1$

i.e., Assume that $\sum_{i=1}^{k}2i=k^2+k$ for some natural number $k$, and show that $\sum_{i=1}^{k+1}2i=\left(k+1\right)^2+\left(k+1\right)$

**Observe:**

$$\sum_{i=1}^{k+1}2i=\underbrace{\sum_{i=1}^{k}2i+2\left(k+1\right)=k^2+k+2\left(k+1\right)}_{\text{by Induction Hypothesis}}$$

$$=k^2+k+2k+2=k^2+k+\left(k+1\right)+\left(k+1\right)=k^2+2k+1+\left(k+1\right)$$

$$=\left(k+1\right)^2+\left(k+1\right)$$

i.e., $\sum_{i=1}^{k+1}2i=\left(k+1\right)^2+\left(k+1\right)$

Hence, $\sum_{i=1}^{n}2i=n^2+n$ for all natural numbers, $n$. ■