Problem List For Test #3 - Solutions

Fall 2018

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1. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then $\lim_{x\to c} (f(x) + g(x)) = L + M$

   Proof. Let the hypotheses be given (i.e., let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$).

   Let $\varepsilon > 0$ be given.

   Since $\lim_{x\to c} f(x) = L$, there exists a $\delta_f = \delta_f(\varepsilon) > 0$ such that
   $0 < |x - c| < \delta_f \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$.

   Similarly, since $\lim_{x\to c} g(x) = M$, there exists a $\delta_g = \delta_g(\varepsilon) > 0$ such that
   $0 < |x - c| < \delta_g \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$.

   Choose $\delta = \delta(\varepsilon) = \underline{\min(\delta_f, \delta_g)}$.

   Observe: If $0 < |x - c| < \delta$ we have:

   $|(f(x) + g(x)) - (L + M)| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M|$
   $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

   i.e., $0 < |x - c| < \delta \Rightarrow |(f(x) + g(x)) - (L + M)| < \varepsilon$.

   Hence, $\lim_{x\to c} (f(x) + g(x)) = L + M$ ■

Scratch Work: For $0 < |x - c| < \delta$, we want $|(f(x) + g(x)) - (L + M)| < \varepsilon$.

Observe: $|(f(x) + g(x)) - (L + M)| = |(f(x) - L) + (g(x) - M)|$

$\leq |f(x) - L| + |g(x) - M|$

IF we could make each of these less than $\frac{\varepsilon}{2}$, then we'd have:

$|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

i.e., $|(f(x) + g(x)) - (L + M)| < \varepsilon$

Since $\lim_{x\to c} f(x) = L$, $\exists \delta_f > 0$ such that $0 < |x - c| < \delta_f \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$

Similarly, $\exists \delta_g > 0$ such that $0 < |x - c| < \delta_g \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$

Choose $\delta = \delta(\varepsilon) = \min(\delta_f, \delta_g)$

2. Theorem: $\lim_{x \to c} k = k$, for any constant k.

   Proof. Let $\varepsilon > 0$ be given.

   Let $\delta = \delta(\varepsilon) > 0$ be given by $\delta = 1$. (We will see that this choice of δ is completely arbitrary.)

   Observe: Given $0 < |x - c| < \delta$, we have:

   $$\underbrace{|k - k|}_{f(x)-L} = 0 < \varepsilon$$

   i.e., Given $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that $0 < |x - c| < \delta \Rightarrow |k - k| < \varepsilon$.

   Hence, $\lim_{x \to c} k = k$ ■

3. Theorem: If $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} kf(x) = kL$, for any constant k.

   Proof. Let the hypothesis be given (i.e., suppose that $\lim_{x \to c} f(x) = L$), and let $\varepsilon > 0$ be given.

   If $k = 0$, then $|kf(x) = kL| = 0 < \varepsilon$, for any value of x. So we'll assume that $k \neq 0$.

   By hypothesis, $\exists \delta = \delta(\varepsilon) > 0$ such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \frac{\varepsilon}{|k|}$

   Observe: Given $0 < |x - c| < \delta$, we have:

   $$|kf(x) - kL| = |k(f(x) - L)| = |k| \underbrace{|f(x) - L|}_{< \frac{\varepsilon}{|k|}} < |k| \frac{\varepsilon}{|k|} = \varepsilon$$

   i.e., Given $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

   Hence, $\lim_{x \to c} kf(x) = kL$ ■

4. Lemma: If $\lim_{x\to c} g(x) = M \neq 0$, then $\exists\delta = \delta(M,c)$ such that

   $|x-c| < \delta \Rightarrow \frac{|M|}{2} < |g(x)| < \frac{3|M|}{2}$

   Proof. Let the hypothesis be given. By hypothesis, $\exists\delta = \delta(M) > 0$ such that

   $|x-c| < \delta \Rightarrow |g(x) - M| < \frac{|M|}{2}$

   By the Triangle Inequality #5, $||g(x)| - |M|| \leq |g(x) - M|$.

   Thus, given $|x-c| < \delta$, we have:

   $||g(x)| - |M|| \leq |g(x) - M| < \frac{|M|}{2}$

   i.e., $||g(x)| - |M|| < \frac{|M|}{2}$

   $\Rightarrow -\frac{|M|}{2} < |g(x)| - |M| < \frac{|M|}{2}$

   $\Rightarrow \frac{|M|}{2} < |g(x)| < \frac{3|M|}{2}$ ■

5. Corollary: If $\lim_{x\to c} g(x) = M \neq 0$, then $\exists\delta = \delta(M,c) > 0$ such that:

   $|x-c| < \delta \Rightarrow \frac{1}{|g(x)|} < \frac{2}{|M|}$

6. Lemma: If $\lim_{x\to c} g(x) = M \neq 0$, then $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$

   Proof. Suppose that $\lim_{x\to c} g(x) = M \neq 0$, and let $\varepsilon > 0$ be given.

   Define $\delta = \delta(\varepsilon, c)$ to be given by $\delta = \underline{\min(\delta_1, \delta_2)}$, where δ_1 is such that

   $|x-c| < \delta_1 \Rightarrow \frac{|M|}{2} < |g(x)| < \frac{3|M|}{2}$

   (the existence of such a δ_1 is guaranteed by a previous lemma),

   and where δ_2 is such that $|x-c| < \delta_2 \Rightarrow |g(x) - M| < \frac{M^2\varepsilon}{2}$.

   (the existence of such a δ_2 by virtue of the fact that $\lim_{x\to c} g(x) = M$).

   Let $|x-c| < \delta$

   Observe: $\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M-g(x)}{Mg(x)}\right| = |g(x) - M| \frac{1}{|Mg(x)|} = |g(x) - M| \frac{1}{|M|}\frac{1}{|g(x)|}$

   $< |g(x) - M| \frac{1}{|M|}\frac{2}{|M|} = |g(x) - M| \frac{2}{M^2} < \frac{M^2\varepsilon}{2}\frac{2}{M^2} = \varepsilon$

   i.e., $|x-c| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \varepsilon$

   Therefore, $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$ ■

Scratchwork: We want $\left|\frac{1}{g(x)}-\frac{1}{M}\right|<\varepsilon$ whenever $|x-c|<\delta$.

$\left|\frac{1}{g(x)}-\frac{1}{M}\right|=\left|\frac{M-g(x)}{Mg(x)}\right|=|g(x)-M|\frac{1}{|Mg(x)|}=|g(x)-M|\frac{1}{|M|}\frac{1}{|g(x)|}$

By a previous lemma, $\exists\delta=\delta(M)>0$ such that:

$\frac{|M|}{2}<|g(x)|<\frac{3|M|}{2}$.

Hence, we can make $\frac{1}{|g(x)|}<\frac{2}{|M|}$ for $|x-c|<\delta$

So that gives us

$\left|\frac{1}{g(x)}-\frac{1}{M}\right|=|g(x)-M|\frac{1}{|M|}\frac{1}{|g(x)|}<|g(x)-M|\frac{1}{|M|}\frac{2}{|M|}=|g(x)-M|\frac{2}{M^2}$

Since $\lim_{x\to c}g(x)=M$, we can make $|g(x)-M|$ less than any positive real number, provided that $|x-c|<\delta$

We want $|g(x)-M|$ sufficiently small to guarantee that in order to make

$\left|\frac{1}{g(x)}-\frac{1}{M}\right|<\varepsilon$

Thus, we'd like to have: $\left|\frac{1}{g(x)}-\frac{1}{M}\right|<|g(x)-M|\frac{2}{M^2}<\varepsilon$

$\Rightarrow|g(x)-M|<\frac{M^2\varepsilon}{2}$

We've imposed two conditions on δ :

First, $|x-c|<\delta\Rightarrow\frac{1}{|g(x)|}<\frac{2}{|M|}$

Second, $|x-c|<\delta\Rightarrow|g(x)-M|<\frac{M^2\varepsilon}{2}$

Our choice of δ will be the smaller of the two.

7. If $\lim_{x\to c}f(x)=L$ and $\lim_{x\to c}g(x)=M\neq0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{L}{M}$

   Proof. Let the hypotheses be given.

   Observe: $\lim_{x\to c}\frac{f(x)}{g(x)}=\underbrace{\lim_{x\to c}\left(f(x)\cdot\frac{1}{g(x)}\right)=\left(\lim_{x\to c}f(x)\right)\left(\lim_{x\to c}\frac{1}{g(x)}\right)}_{\text{Because the limit of a product equals the product of the limits}}=L\cdot\frac{1}{M}=\frac{L}{M}$

   i.e., $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{L}{M}$ ■

8. **Prove:** The sequence $\left\{\frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \ldots, \frac{2n-1}{2n+1}, \ldots\right\}$ converges to $L = 1$.

Proof. Let $\varepsilon > 0$ be given.

(We must show that $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow \underbrace{\left|\dfrac{2n-1}{2n+1} - 1\right|}_{|a_n - L|} < \varepsilon$)

Let $N = N(\varepsilon)$ be given by $\left\lceil \frac{1}{\varepsilon} \right\rceil$

Let $n > N$

Observe:

$$\underbrace{\left|\dfrac{2n-1}{2n+1} - 1\right|}_{|a_n - L|} = \left|\frac{2n-1}{2n+1} - \frac{2n+1}{2n+1}\right| = \left|\frac{-2}{2n+1}\right| = \frac{2}{2n+1} < \frac{2}{2N+1} < \frac{2}{2N} = \frac{1}{N} \leq \frac{1}{\left(\frac{1}{\varepsilon}\right)} = \varepsilon$$

i.e., $n > N \Rightarrow \underbrace{\left|\dfrac{2n-1}{2n+1} - 1\right|}_{|a_n - L|} < \varepsilon$

Hence, The sequence $\left\{\frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \ldots, \frac{2n-1}{2n+1}, \ldots\right\}$ converges to $L = 1$.

(i.e., $\lim_{n\to\infty} \frac{2n-1}{2n+1} = 1$) ■

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Scratchwork: Given $n > N$, we want $\left|\frac{2n-1}{2n+1} - 1\right| < \varepsilon$

$\left|\frac{2n-1}{2n+1} - 1\right| = \left|\frac{2n-1}{2n+1} - \frac{2n+1}{2n+1}\right| = \left|\frac{-2}{2n+1}\right| = \frac{2}{2n+1} < \frac{2}{2N+1} < \frac{2}{2N} = \frac{1}{N}$

i.e., $\left|\frac{2n-1}{2n+1} - 1\right| < \frac{1}{N}$

We can make $\left|\frac{2n-1}{2n+1} - 1\right| < \varepsilon$ by doing the following:

$\left|\frac{2n-1}{2n+1} - 1\right| < \frac{1}{N} \leq \varepsilon$

This means that $\frac{1}{\varepsilon} \leq N$

i.e., $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$ (The least integer greater than or equal to $\frac{1}{\varepsilon}$.)

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9. **Prove:** The sequence $\left\{\frac{1}{5}, \frac{2}{8}, \frac{3}{11}, \ldots, \frac{n}{3n+2}, \ldots\right\}$ converges to $L = \frac{1}{3}$.

10. **Prove:** The sequence $\left\{\frac{1}{6}, \frac{7}{11}, \frac{13}{16}, \ldots, \frac{6n-5}{5n+1}, \ldots\right\}$ converges to $L = \frac{6}{5}$.

11. **Prove:** The sequence $\left\{1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots\right\}$ converges to $L = 0$.

12. **Prove:** The sequence $\left\{\frac{3}{\sqrt{11}}, \frac{3}{\sqrt{14}}, \frac{3}{\sqrt{19}}, \ldots, \frac{3}{\sqrt{n^2+10}}, \ldots\right\}$ converges to $L = 0$.

13. Problems like 8-12

14. Prove: The sequence $\{a_n\}_{n=1}^{\infty}$ converges to L if and only if every open interval of the form $(L-\varepsilon, L+\varepsilon)$ contains all but finitely many terms of the sequence.

Proof.

$\{a_n\}_{n=1}^{\infty}$ converges to $L \Rightarrow$ every open interval of the form $(L-\varepsilon, L+\varepsilon)$ contains all but finitely many terms of the sequence.

Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L, and let $\varepsilon > 0$ be given.

Then $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow |a_n - L| < \varepsilon$

$\Rightarrow -\varepsilon < a_n - L < \varepsilon$

$\Rightarrow L - \varepsilon < a_n < L + \varepsilon$

i.e., $n > N \Rightarrow a_n \in (L-\varepsilon, L+\varepsilon)$.

Thus the only terms of the sequence that may not be contained in the interval $(L-\varepsilon, L+\varepsilon)$ are $\{a_1, a_2, a_3, \ldots, a_N\}$. Therefore, every open interval of the form $(L-\varepsilon, L+\varepsilon)$ contains all but finitely many terms of the sequence.

Every open interval of the form $(L-\varepsilon, L+\varepsilon)$ contains all but finitely many terms of $\{a_n\}_{n=1}^{\infty} \Rightarrow \{a_n\}_{n=1}^{\infty}$ converges to L.

Let $\varepsilon > 0$ be given and suppose that every open interval of the form $(L-\varepsilon, L+\varepsilon)$ contains all but finitely many terms of the seuqence $\{a_n\}_{n=1}^{\infty}$.

Then there are only finitely many terms of the sequence that are *not* contained in the interval $(L-\varepsilon, L+\varepsilon)$.

If all terms of the sequence are contained in the interval $(L-\varepsilon, L+\varepsilon)$, then let $N = 1$.

Otherwise, let N be the largest natural number, such that a_N is not contained in the interval $(L-\varepsilon, L+\varepsilon)$.

Then $n > N \Rightarrow a_n \in (L-\varepsilon, L+\varepsilon)$

$\Rightarrow L - \varepsilon < a_n < L + \varepsilon$

$\Rightarrow -\varepsilon < a_n - L < \varepsilon$

$\Rightarrow |a_n - L| < \varepsilon$

i.e., $\exists N \in \mathbf{N}$ such that $n > N \Rightarrow |a_n - L| < \varepsilon$.

Thus, $\{a_n\}_{n=1}^{\infty}$ converges to L. ■

15. Prove: If a sequence converges (i.e., if it has a limit, L, where L is a real number), then the limit is unique.

Proof. (By contradiction) Suppose, for the sake of deriving a contradiction, that the sequence $\{a_n\}_{n=1}^{\infty}$ converges and has at least two distinct limits, L_1 and L_2.

Without loss of generality, assume that $L_1 < L_2$. Therefore, there exists an $\varepsilon > 0$ such that $L_2 - L_1 = \varepsilon$.

Since $\{a_n\}_{n=1}^{\infty}$ converges to L_1, $\exists N_1$ such that $n > N_1 \Rightarrow |a_n - L_1| < \frac{\varepsilon}{3}$.

Since $\{a_n\}_{n=1}^{\infty}$ converges to L_2, $\exists N_2$ such that $n > N_2 \Rightarrow |a_n - L_2| < \frac{\varepsilon}{3}$.

Let $N = \max(N_1, N_2)$

Then for $n > N$, we have:

$|a_n - L_1| < \frac{\varepsilon}{3}$ and $|a_n - L_2| < \frac{\varepsilon}{3}$

Consequently, for $n > N$ we have: $\varepsilon = |L_1 - L_2| = |L_1 - a_n + a_n - L_2|$

$\leq |L_1 - a_n| + |a_n - L_2| = |L_1 - a_n| + |L_2 - a_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$

i.e., $\varepsilon < \frac{2\varepsilon}{3}$, a contradiction.

Since the assumption that $\{a_n\}_{n=1}^{\infty}$ has at least two distinct limits, leads to a contradiction, the assumption must be false. Therefore the limit of a convergent sequence is unique. ■

16. **Alternate Proof:** If a sequence converges (i.e., if it has a limit, L, where L is a real number), then the limit is unique.

Proof. Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L_1 and L_2.

Let $\varepsilon > 0$ be given.

Since $\{a_n\}_{n=1}^{\infty}$ converges to L_1, $\exists\ N_1 = N_1(\varepsilon) \in \mathbf{N}$ such that $n > N_1 \Rightarrow |a_n - L_1| < \frac{\varepsilon}{2}$.

Similarly, since $\{a_n\}_{n=1}^{\infty}$ converges to L_2, $\exists\ N_2 = N_2(\varepsilon) \in \mathbf{N}$ such that $n > N_2 \Rightarrow |a_n - L_2| < \frac{\varepsilon}{2}$.

Let $N = \max(N_1, N_2)$.

Then given $n > N$, we have:

$|L_1 - L_2| = |L_1 - a_n + a_n - L_2| = |(L_1 - a_n) + (a_n - L_2)| \leq |L_1 - a_n| + |a_n - L_2| = |L_1 - a_n| + |L_2 - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
i.e., $\forall \varepsilon \in \mathbf{R}$ with $\varepsilon > 0$, $|L_1 - L_2| < \varepsilon$

$\Rightarrow |L_1 - L_2| = 0$

$\Rightarrow L_1 = L_2$ ■

17. Prove: Every convergent sequence is bounded.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit L and let $\varepsilon = 1$ be given.

Then, by a previous theorem, the open interval $(L-1, L+1)$ contains all but finitely many terms of the sequence.

If all terms of the sequence are contained in the interval $(L-1, L+1)$, then $L - 1 \leq a_n \leq L + 1\ \forall n \in \mathbf{N}$, and consequently, the sequence is bounded.

Otherwise, let $\{a_{n_1}, a_{n_2}, \ldots, a_{n_k}\}$ be the terms of the sequence which are not contained in the interval $(L-1, L+1)$. Since there are only finitely many of these terms, there must be a largest and a smallest term. Let M and m be the values of the largest and smallest terms, respectively, of those terms of the sequence that are not contained in the interval $(L-1, L+1)$.

Then, $\min(m, L-1) \leq a_n \leq \max(M, L+1)\ \forall n \in \mathbf{N}$.

Thus, the sequence is bounded. ■

18. **Prove:** if the sequence $\{a_n\}_{n=1}^{\infty}$ converges to zero and the sequence $\{b_n\}_{n=1}^{\infty}$ is bounded, then the sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges to zero.

Proof. Let the hypotheses be given. Specifically, suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to zero and that the sequence $\{b_n\}_{n=1}^{\infty}$ is bounded.

Since the sequence $\{b_n\}_{n=1}^{\infty}$ is bounded, $\exists\ M \in \mathbb{R}$ such that $|b_n| < M$, $\forall n \in \mathbb{N}$.

(Such an M has to exist. If no such number M existed, then this would mean that the terms of $\{b_n\}_{n=1}^{\infty}$ get large (in magnitude) without bound.)

Since the sequence $\{a_n\}_{n=1}^{\infty}$ converges to zero, given any $\varepsilon > 0$, we can choose $N = N(\varepsilon) \in \mathbb{N}$ such that $n > N \Rightarrow |a_n - 0| < \varepsilon$

More generally, the fact that $\{a_n\}_{n=1}^{\infty}$ converges to zero, means that:

We can choose $N = N(\varepsilon) \in \mathbb{N}$ such that $n > N \Rightarrow |a_n - 0| <$ (any positive real number that we choose.)

Based on our scratchwork below, given $\varepsilon > 0$, we will choose $N = N(\varepsilon) \in \mathbb{N}$ such that $n > N \Rightarrow |a_n - 0| < \frac{\varepsilon}{M}$.

Observe: Given $\varepsilon > 0$, and $N = N(\varepsilon) \in \mathbb{N}$ such that $n > N \Rightarrow |a_n - 0| < \frac{\varepsilon}{M}$, we have:

$$|a_n b_n - 0| = |a_n b_n| = \underbrace{|a_n|}_{<\frac{\varepsilon}{M}} \underbrace{|b_n|}_{<M} < \left(\frac{\varepsilon}{M}\right)(M) = \varepsilon$$

i.e., Given $\varepsilon > 0$, $\exists\ N = N(\varepsilon) \in \mathbb{N}$ such that $n > N \Rightarrow |a_n b_n - 0| < \varepsilon$.

Hence, $\{a_n b_n\}_{n=1}^{\infty}$ converges to zero. ■

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Scratchwork:

To show that $\{a_n b_n\}_{n=1}^{\infty}$ converges to zero, we must be able to find $N \in \mathbb{N}$ such that $n > N \Rightarrow |a_n b_n - 0| < \varepsilon$

(i.e., $n > N \Rightarrow |a_n b_n| < \varepsilon$)

Observe: $|a_n b_n| = |a_n| |b_n| \leq |a_n| M < \varepsilon$

But $|a_n| M < \varepsilon \Rightarrow |a_n| < \frac{\varepsilon}{M}$

So we choose $N \in \mathbb{N}$ such that $n > N \Rightarrow |a_n - 0| < \frac{\varepsilon}{M}$

End Scratchwork

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19. Let S be the set of points in the interval $(0,1)$ having denominator 2^n for $n = 1, 2, 3, \ldots$.

    (a) Does S have any limit points? If so, name them, and prove that they are limit points.

20. ~

    (a) Prove that every point in the interval $(0,1)$ is a limit point of the interval $(0,1)$.

    (b) Are there any limit points of the interval $(0,1)$ which are not contained in the interval? Justify your answer.

21. Given the set $\{1,\ 1.1,\ 0.9,\ 1.01,\ 0.99,\ 1.001,\ 0.999, \ldots\}$

    (a) Is the set bounded?

    (b) Does the set have a l.u.b. or a g.l.b? If so, what are they?

    (c) Does the set have any limit points? If so, what are they?

22. Given the set $\{-0.9,\ 0.9,\ -0.99,\ 0.99,\ -0.999,\ 0.999, \ldots\}$

    (a) Is the set bounded?

    (b) Does the set have a l.u.b. or a g.l.b? If so, what are they?

    (c) Does the set have any limit points? If so, what are they?

23. State and Prove the Monotone Convergence Theorem.

Remark 1 *(This is sometimes called the Bounded Convergence Theorem)*

Every monotone increasing sequence, bounded above, converges. (Similarly, every monotone decreasing sequence, bounded below, converges.)

Proof. We will consider the case in which the sequence is monotone increasing, bounded above. The other case is similar.

Let $\{a_n\}_{n=1}^{\infty}$ be monotone increasing and bounded above. Since the sequence is monotone increasing, it is bounded below (in fact, a_1 is the greatest lower bound).

Since $\{a_n\}_{n=1}^{\infty}$ is bounded, the least upper bound axiom of real numbers guarantees that $\{a_n\}_{n=1}^{\infty}$ has a least upper bound. We'll call it U.

We intend to show that $\{a_n\}_{n=1}^{\infty}$ converges to U.

Let $\varepsilon > 0$ be given.

Since U is the least upper bound for $\{a_n\}_{n=1}^{\infty}$, $\exists N \in \mathbf{N}$ such that $a_N > U - \varepsilon$. (Otherwise, $U-\varepsilon$ would be an upper bound of $\{a_n\}_{n=1}^{\infty}$ that is less than U, contradicting the fact that U is the *least* upper bound.)

Since $\{a_n\}_{n=1}^{\infty}$ is increasing, $n > N \Rightarrow a_n > a_N$.

Thus, $\forall n > N$, we have: $U - \varepsilon < a_N < a_n < U$

i.e., $\forall n > N$, $U - \varepsilon < a_n < U$

$\Rightarrow \forall n > N$, $U - \varepsilon < a_n < U + \varepsilon$

$\Rightarrow \forall \varepsilon > 0$, all but finitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$ are contained in the interval $(U - \varepsilon, U + \varepsilon)$.

Thus, $\{a_n\}_{n=1}^{\infty}$ converges to U. ■

24. State and Prove the Nested Interval Theorem.

Suppose that $\{[a_n, b_n]\}_{n=1}^{\infty}$ is a sequence of closed, nested intervals. (i.e., suppose that $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \ldots \supseteq [a_n, b_n] \supseteq \ldots$ and that $\lim_{n\to\infty}(b_n - a_n) = 0$.)

Then $\exists!\ p \in \mathbf{R}$ such that $p \in \cap_{n=1}^{\infty} [a_n, b_n]$.

Proof. Observe: $\{a_n\}_{n=1}^{\infty}$ is monotone increasing and is bounded above by b_1. By the monotone convergence theorem, $\{a_n\}_{n=1}^{\infty}$ converges to some real number a, where $a = l.u.b.\ \{a_n\}_{n=1}^{\infty}$.

Similarly, $\{b_n\}_{n=1}^{\infty}$ converges to some real number b, where $b = g.l.b.\ \{b_n\}_{n=1}^{\infty}$.

It is our strategy to show that $a = b$. We'll name the common value p. (i.e. $p = a = b$.)

Thus, since $p = l.u.b.\ \{a_n\}_{n=1}^{\infty} = g.l.b.\ \{b_n\}_{n=1}^{\infty}$, it will follow that
$a_n \leq p \leq b_n\ \forall n \in \mathbf{N}$. (i.e., $p \in [a_n, b_n]\ \forall n \in \mathbf{N}$.),
from which it will follow that $p \in \cap_{n=1}^{\infty} [a_n, b_n]$.

Let $\varepsilon > 0$.

In order to show that $a = b$, we need to make some preliminary observations.

Since $\{a_n\}_{n=1}^{\infty}$ converges to a, $\exists N_a \in \mathbf{N}$ such that $n > N_a \Rightarrow |a_n - a| < \frac{\varepsilon}{3}$

Since $\{b_n\}_{n=1}^{\infty}$ converges to b, $\exists N_b \in \mathbf{N}$ such that $n > N_b \Rightarrow |b_n - b| < \frac{\varepsilon}{3}$

Finally, since $\lim_{n\to\infty}(b_n - a_n) = 0$, $\exists N_{ab} \in \mathbf{N}$ such that $n > N_{ab} \Rightarrow |b_n - a_n| < \frac{\varepsilon}{3}$

Let $N = \max(N_a, N_b, N_{ab})$.

Observe: For $n > N$, we have:

$|b - a| = |b - b_n + b_n - a_n + a_n - a| \leq |b - b_n| + |b_n - a_n| + |a_n - a| =$

$|b_n - b| + |b_n - a_n| + |a_n - a| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

i.e., $\forall \varepsilon > 0$, $|b - a| < \varepsilon$.

Hence, $b - a = 0 \Rightarrow a = b$.

Furthermore, p is unique. For given any $q \in \mathbf{R}$, with $q \neq p$, we have either $q < p$ or $q > p$.

If $q < p$, then $\exists k \in \mathbf{N}$ such that $a_k > q$. (Otherwise, q would be an upper bound of

$\{a_n\}_{n=1}^{\infty}$ that is less than p, contradicting the fact that p is the *least* upper bound of

$\{a_n\}_{n=1}^{\infty}$).

Consequently, $q \notin [a_k, b_k] \Rightarrow q \notin \cap_{n=1}^{\infty} [a_n, b_n]$.

Similarly, if $q > p$, then $q \notin \cap_{n=1}^{\infty} [a_n, b_n]$. ■

25. State and Prove the Bolzano-Weierstrass Theorem.

Every bounded, infinite set of real numbers has a limit point.

Let S be a bounded, infinite set of real numbers. Then S has a g.l.b. and a l.u.b., call them a and b, respectively.

Consider the midpoint $\frac{a+b}{2}$ of the interval. Note that at least one of the intervals, $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$, must contain infinitely many elements of the set S. (Otherwise, both intervals contain only finitely many elements of S, and hence, their union, $[a, b]$ must contain only finitely many elements of set S, contradicting the fact that $[a, b]$ contains S, which is infinite.)

Select an interval (either $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$) which contains infinitely many elements of S, and rename it $[a_1, b_1]$.

Observe that $[a, b] \supseteq [a_1, b_1]$, and that $(b_1 - a_1) = \frac{b-a}{2}$.

Consider the intervals, $\left[a_1, \frac{a_1+b_1}{2}\right]$ and $\left[\frac{a_1+b_1}{2}, b_1\right]$. At least one of these intervals contains infinitely many elements of S. Choose one that does, and rename it $[a_2, b_2]$.

Observe that $[a, b] \supseteq [a_1, b_1] \supseteq [a_2, b_2]$, and that $(b_2 - a_2) = \frac{b_1 - a_1}{2} = \frac{b-a}{2^2}$.

Continuing inductively, we generate a sequence of closed, nested intervals:

$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \ldots \supseteq [a_n, b_n] \supseteq \ldots$ where $(b_n - a_n) = \frac{b-a}{2^n}$, and hence, $\lim_{n\to\infty} (b_n - a_n) = 0$.

Thus, $\exists!\ p \in \mathbf{R}$ such that $p \in \cap_{n=1}^{\infty} [a_n, b_n]$.

We claim that p is a limit point of S.

To show this, we must show that $\forall \varepsilon > 0$, the interval $(p - \varepsilon, p + \varepsilon)$ contains infinitely many points of the set S.

So let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty} (b_n - a_n) = 0$, $\exists n \in \mathbf{N}$ such that $(b_n - a_n) < \varepsilon$

Since $a_n < p < b_n$, we have:

$b_n - p < b_n - a_n < \varepsilon$

i.e., $b_n - p < \varepsilon \Rightarrow b_n < p + \varepsilon$

Similarly, $p - a_n < b_n - a_n < \varepsilon$

i.e., $p - a_n < \varepsilon \Rightarrow p - \varepsilon < a_n$

Thus, we have: $p - \varepsilon < a_n < b_n < p + \varepsilon$

Hence, interval $[a_n, b_n]$ is contained in the interval $(p - \varepsilon, p + \varepsilon)$.

Since (by hypothesis) $[a_n, b_n]$ contains infinitely many elements of the set S, the interval $(p - \varepsilon, p + \varepsilon)$ contains infinitely many elements of the set S also.

Therefore, p is a limit point of S. ■

26. Prove that the sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\}$ has a limit point.

Proof. Observe: $L = 0$ is a lower bound of the sequence and $U = 1$ is an upper bound. Therefore, by the Bolzano-Weierstrass Theorem, the sequence has a limit point. ■

27. Prove that the sequence $\left\{0, -\frac{1}{2}, \frac{3}{4}, -\frac{7}{8}, \dots\right\}$ where $a_n = \begin{cases} 1 - \frac{1}{2^{n-1}} & \text{for } n \text{ odd} \\ -1 + \frac{1}{2^{n-1}} & \text{for } n \text{ even} \end{cases}$ has a limit point.

Proof. Observe: $L = -1$ is a lower bound of the sequence and $U = 1$ is an upper bound. Therefore, by the Bolzano-Weierstrass Theorem, the sequence has a limit point. ■

28. Problems like 26 and 27.

29. Prove: Every Bounded sequence has a convergent subsequence.

30. If $f(x)$ is a continuous function and $\{a_n\}_{n=1}^{\infty}$ is a sequence which converges to a, then $\{f(a_n)\}_{n=1}^{\infty}$ is a sequence which converges to $f(a)$.

31. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L if and only if every subsequence converges to L.

32. Definition: *Cauchy sequence*

33. Every convergent sequence is a Cauchy sequence.

Proof. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence, and let L be the limit of the sequence.

Let $\varepsilon > 0$ be given.

Since $\{a_n\}_{n=1}^{\infty}$ converges to L, $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow |a_n - L| < \frac{\varepsilon}{2}$.

Thus, for $m, n > N$, we have $|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| =$

$|a_m - L| + |a_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

i.e., $m, n > N \Rightarrow |a_m - a_n| < \varepsilon$.

Thus, $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. ∎

Scratch Work: For $m, n > N$, we want $|a_m - a_n| < \varepsilon$.

In order to use the fact that $\{a_n\}_{n=1}^{\infty}$ converges to L, we somehow must work L into the inequality $|a_m - a_n| < \varepsilon$.

So, we'll use the old "add and subtract" trick.

$\Rightarrow |a_m - a_n| < \varepsilon \Rightarrow |a_m - L + L - a_n| < \varepsilon \Rightarrow |a_m - L + L - a_n| \leq$

$|a_m - L| + |L - a_n| < \varepsilon \Rightarrow |a_m - L| + |a_n - L| < \varepsilon$

Observe: if we make $|a_m - L| < \frac{\varepsilon}{2}$ and $|a_n - L| < \frac{\varepsilon}{2}$, then we'll have $|a_m - L + L - a_n| \leq$

$|a_m - L| + |L - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus, we choose $N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow |a_n - L| < \frac{\varepsilon}{2}$.

34. Every Cauchy Sequence has a limit point.

Proof. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Then $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $m, n > N \Rightarrow |a_m - a_n| < 1$. (Here, 1 plays the role of ε.)

Thus, for all $n > N$ we have: $|a_n - a_{N+1}| < 1$

$\Rightarrow -1 < a_n - a_{N+1} < 1$

$\Rightarrow a_{N+1} - 1 < a_n < a_{N+1} + 1$

The point of this, is that all terms of the sequence after the N^{th} term are *bounded* by $a_{N+1} - 1$ and $a_{N+1} + 1$.

Hence, $L = \min(a_1, a_2, a_3, \ldots, a_N, a_{N+1} - 1)$ is a lower bound of $\{a_n\}_{n=1}^{\infty}$

and $U = \max(a_1, a_2, a_3, \ldots, a_N, a_{N+1} + 1)$ is an upper bound of $\{a_n\}_{n=1}^{\infty}$

Thus the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.

By the Bolzano-Weierstrass Theorem, the sequence has a limit point. ■

35. Every Cauchy Sequence is convergent.

(i.e., Every Cauchy sequence converges to a limit, L.)

Proof. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Let $\varepsilon > 0$ be given.

Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $m, n > N \Rightarrow |a_m - a_n| < \frac{\varepsilon}{2}$

Also, since the sequence is Cauchy, it has at least one limit point.

Let L be a limit point of $\{a_n\}_{n=1}^{\infty}$.

Since L is a limit point of $\{a_n\}_{n=1}^{\infty}$, every open interval of the form $\underline{\left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)}$ contains infinitely many terms of the sequence.

Consequently, $\exists k > N$ such that $a_k \in \underline{\left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)}$

This is equivalent to saying that $L - \frac{\varepsilon}{2} < a_k < L + \frac{\varepsilon}{2}$

$\Leftrightarrow -\frac{\varepsilon}{2} < a_k - L < \frac{\varepsilon}{2}$

$\Leftrightarrow |a_k - L| < \frac{\varepsilon}{2}$

i.e., $\exists k > \mathbf{N}$ such that $|a_k - L| < \frac{\varepsilon}{2}$.

Observe: $\forall n > N$, we have:

$|a_n - L| = |a_n - a_k + a_k - L| \leq |a_n - a_k| + |a_k - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

i.e., $n > N \Rightarrow |a_n - L| < \varepsilon$.

Hence, $\{a_n\}_{n=1}^{\infty}$ converges to L. ■

Remark 2 *Since the limit of a convergence sequence is unique, L is unique. Thus the theorem (problem 34) which tells us that: "Every Cauchy Sequence has a limit point," can be modified to say that: "Every Cauchy Sequence has exactly one limit point."*

Scratch Work: We want $|a_n - L| < \varepsilon$.

Somehow, we have to relate $|a_n - L|$ to the fact that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since L is a limit of the sequence $\{a_n\}_{n=1}^{\infty}$, every open interval of the form $(L-\delta, L+\delta)$ contains infinitely many terms of the sequence. Thus, there are infinitely many terms of the sequence within δ units of L (i.e., $|a_n - L| < \delta$ for infinitely many terms a_n of the seuqence.) However, this fact alone doesn't guarantee that $|a_n - L| < \delta$ for ALL terms a_n for which $n > N$ for some $N \in \mathbf{N}$.

Thus, for some k for which $|a_k - L| < \delta$, we must find a way to relate "ALL" terms of the sequence to a_k, and thus to L.

How do we do this?

Recall: Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, given any $\delta > 0$, $\exists N$ such that $m, n > N \Rightarrow |a_m - a_n| < \delta$.

Having selected N, recall that we can find a $k > N$ such that $|a_k - L| < \delta$.

Thus, we have for ALL $n > N$:

$|a_n - L| = |a_n - a_k + a_k - L| \leq |a_n - a_k| + |a_k - L| < \delta + \delta = 2\delta$.

i.e., $|a_n - L| < 2\delta$

Since we want $|a_n - L| < \varepsilon$, we let $\varepsilon = 2\delta$ (or $\delta = \frac{\varepsilon}{2}$)

In retrospect, we require that $\exists N$ such that $m, n > N \Rightarrow |a_m - a_n| < \frac{\varepsilon}{2}$

and that every open interval of the form $\left(L - \frac{\varepsilon}{2}., L + \frac{\varepsilon}{2}\right)$ contains infinitely many terms of the sequence.

36. (Corollary) A sequence $\{a_n\}_{n=1}^{\infty}$ converges if and only if it is a Cauchy sequence.