# MTH 4424 Homework and Lecture Notes

Fall 2014

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**Instructions:** Prove or Disprove the following. In the case where the claim is false, provide a counter-example.

1. The square of an odd natural number is odd.

   (i.e., If $n \in \mathbb{N}$ and $n$ is odd, then $n^2$ is odd also.)

   **Proof.** Let $n$ be an odd natural number. Then $n$ can be represented as $n = 2k + 1$ for some natural number $k$.

   Observe: $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\underbrace{(2k^2 + 2k)}_{m} + 1 = 2m + 1$

   i.e., $n^2$ has the form $2m + 1$. Hence, $n^2$ is odd. ■

2. If $m \in \mathbb{N}$ and $m^2$ is even, then $m$ is even also.

   **Proof.** This is just the contrapositive of the preceding theorem. Hence, it is true. ■

3. The square of an even natural number is even.

   (i.e., If $n \in \mathbb{N}$ and $n$ is even, then $n^2$ is even also.)

   **Proof.** Let $n$ be an even natural number. Then $n$ can be represented as $n = 2k$ for some natural number $k$.

   Observe: $n^2 = (2k)^2 = 4k^2 = 2\underbrace{(2k^2)}_{m} = 2m$

   i.e., $n^2$ has the form $2m$. Hence, $n^2$ is even. ■

4. If $m \in \mathbb{N}$ and $m^2$ is odd, then $m$ is odd also.

   **Proof.** This is just the contrapositive of the preceding theorem. Hence, it is true. ■

5. Consequently, we have the following theorem:

   If $m \in \mathbb{N}$, then $m^2$ is even if and only if $m$ is even, and $m^2$ is odd if and only if $m$ is odd.

6. $\sqrt{2}$ is an irrational number.

   **Proof.** Suppose, for the sake of deriving a contradiction, that $\sqrt{2}$ is rational.

   Then $\exists\ m, n \in \mathbb{N}$ such that $\sqrt{2} = \frac{m}{n}$.

   Without loss of generality, we can assume that $m$ and $n$ are relatively prime.*

   Thus, we have: $\sqrt{2} = \frac{m}{n}$.

   $\Rightarrow 2 = \frac{m^2}{n^2}$

   $\Rightarrow 2n^2 = m^2$

   $\Rightarrow m^2$ is even

   $\Rightarrow m$ is even

   $\Rightarrow \exists\ k \in \mathbb{N}$ such that $m = 2k$.

   Thus, $2n^2 = m^2 = (2k)^2 = 4k^2$

   i.e., $2n^2 = 4k^2$

   $\Rightarrow n^2 = 2k^2$

   $\Rightarrow n^2$ is even

   $\Rightarrow n$ is even

   i.e., $m$ and $n$ are both even, and consequently, $m$ and $n$ both have a factor of 2.

   This contradicts the assumption that $m$ and $n$ are relatively prime.

   Since the assumption that $\sqrt{2}$ is rational leads to a contradiction, the assumption must be false.

   Hence, $\sqrt{2}$ is irrational. ■

   *If $m$ and $n$ are not relatively prime, then let $d$ be the greatest common divisor of $m$ and $n$. There exist relatively prime integers $m_1$ and $n_1$ such that $m = dm_1$ and $n = dn_1$. Thus we can write $\sqrt{2} = \frac{m}{n} = \frac{dm_1}{dn_1} = \frac{m_1}{n_1}$, and $\sqrt{2} = \frac{m_1}{n_1}$ is written as the quotient of relatively prime integers.

7. The sum or difference of rational numbers is rational.

   **Proof.** Let $x, y \in \mathbf{Q}$.

   Then $\exists\ m, n, r, s \in \mathbf{Z}$ with $n, s \neq 0$ such that $x = \frac{m}{n}$ and $y = \frac{r}{s}$.

   Observe: $x \pm y = \frac{m}{n} \pm \frac{r}{s} = \frac{ms \pm nr}{ns}$.

   Since integers are closed under addition, subtraction, and multiplication, $x \pm y = \frac{ms \pm nr}{ns}$ is the quotient of integers, hence rational. ■

8. The product of rational numbers is rational.

   **Proof.** Let $x, y \in \mathbf{Q}$.

   Then $\exists\, m, n, r, s \in \mathbf{Z}$ with $n, s \neq 0$ such that $x = \frac{m}{n}$ and $y = \frac{r}{s}$.

   Observe: $x \cdot y = \frac{m}{n} \cdot \frac{r}{s} = \frac{mr}{ns}$.

   Since integers are closed under multiplication, $x \cdot y = \frac{mr}{ns}$ is the quotient of integers, hence rational. ■

9. The quotient of rational numbers is rational.

   This is FALSE.

   **Proof.** As a counter-example, consider $x = 1$ and $y = 0$.

   Observe: $x, y \in \mathbf{Q}$, and yet $\frac{x}{y}$ is undefined (hence, not rational). ■

10. The quotient of rational numbers is rational, provided that the divisor is non-zero.

    **Proof.** Let $x, y \in \mathbf{Q}$, with $y \neq 0$.

    Then $\exists\, m, n, p, q \in \mathbf{Z}$ with $n, p, q \neq 0$ such that $x = \frac{m}{n}$ and $y = \frac{p}{q}$.

    Observe: $\frac{x}{y} = \frac{\left(\frac{m}{n}\right)}{\left(\frac{p}{q}\right)} = \frac{m}{n} \cdot \frac{q}{p} = \frac{mq}{np}$

    Since integers are closed under multiplication, $mq$ and $np$ are integers. Furthermore, $np$ is nonzero, by the Zero Divisor Property. Therefore, $\frac{x}{y} = \frac{mq}{np}$ is the quotient of integers, hence rational. ■

11. The sum (or difference) of a rational and an irrational is irrational.

    **Proof.** Let $x \in \mathbf{Q}$, and $y \in \mathbf{Q}^c$.

    Suppose, for the sake of deriving a contradiction, that $x + y = z$, where $z \in \mathbf{Q}$.

    Then $y = \underbrace{z}_{\in \mathbf{Q}} - \underbrace{x}_{\in \mathbf{Q}} \Rightarrow y$ is rational, since it is the difference of rationals.

    This contradicts the fact that $y \in \mathbf{Q}^c$.

    Since the assumption that $z \in \mathbf{Q}$ leads to a contradiction, it must be the case that $z \in \mathbf{Q}^c$.

    Hence, the sum of a rational $x$ and an irrational $y$ is the irrational $z$.

    Similarly, the difference of a rational and an irrational is irrational. ■

12. The product or quotient of a rational number and an irrational number is irrational.

    This is FALSE.

    **Proof.** As a counter-example, consider $x = 0$ and let $y$ be any irrational number. Then $x \cdot y = 0$ is the product of a rational and an irrational, and yet it is rational.

    Similarly, $\frac{0}{y} = 0$ is the quotient of a rational and an irrational, and yet it is rational. ■

13. The product or quotient of a *non-zero* rational number and an irrational number is irrational.

**Proof.** Let $x \in \mathbf{Q}$, and $y \in \mathbf{Q}^c$.

Suppose, for the sake of deriving a contradiction, that $x \cdot y = z,$ where $z \in \mathbf{Q}$.

Then $y = \frac{z}{x}$ is rational, since it is the quotient of rationals.

This contradicts the fact that $y \in \mathbf{Q}^c$.

Since the assumption that $z \in \mathbf{Q}$ leads to a contradiction, it must be the case that $z \in \mathbf{Q}^c$.

Hence, the product of a *non-zero* rational $x$ and an irrational $y$ is irrational $z$.

Regarding a quotient, suppose, for the sake of contradiction, that $\frac{x}{y} = z,$ where $z \in \mathbf{Q}$.

Then $y = \frac{x}{z}$ is rational, since it is the quotient of rationals.

This contradicts the fact that $y \in \mathbf{Q}^c$.

Since the assumption that $z \in \mathbf{Q}$ leads to a contradiction, it must be the case that $z \in \mathbf{Q}^c$.

Similarly, since the quotient $\frac{x}{y}$ is irrational, its reciprocal $\frac{y}{x}$ also has to be irrational - otherwise, it would be the quotient of integers, and hence $\frac{x}{y}$ would be the quotient of rationals, which we know to be false. ■

14. The sum or difference of two irrational numbers may or may not be irrational.

**Proof.** This true.

First, we will show that the sum or difference of two irrational numbers can be rational.

Observe: $\sqrt{2} + \left(-\sqrt{2}\right) = 0$ is the sum of irrationals, and the sum is rational.

Similarly, $\sqrt{2} - \sqrt{2} = 0$ is the difference of irrationals, and the difference is rational.

Next, we will show that the sum or difference of two irrational numbers can be irrational.

Observe: $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ is the sum or irrationals, and yet it is irrational, as it is the product of a nonzero rational and an irrational.

Hence, the sum of two irrationals may be irrational.

Similarly, $2\sqrt{2} - \sqrt{2} = \sqrt{2}$ is the difference of irrationals, and this difference is irrational. ■

(a) **Alternatively:**
To show that the sum of two irrationals may be rational, consider $x = 0.101001000100001\ldots$ and $y = 0.010110111011110\ldots$

**Observe:** $x, y \in \mathbf{Q}^c$ and yet, $x + y = 0.11111111111111\ldots \in \mathbf{Q}$

To show that the sum of two irrationals may be irrational, consider $x = y = 0.101001000100001\ldots$

**Observe:** $x, y \in \mathbf{Q}^c$ and yet, $x + y = 0.202002000200002\ldots \in \mathbf{Q}^c$

**As another example,** consider $x = y = 0.101001000100001\ldots$ and $y = 0.020220222022220\ldots$

**Observe:** $x, y \in \mathbf{Q}^c$ and yet, $x + y = 0.121221222122221\ldots \in \mathbf{Q}^c$ ■

15. The product or quotient of two irrational numbers may or may not be irrational.

**Proof.** To show that the product or quotient of two irrational numbers may be rational, observe that $\sqrt{2} \cdot \sqrt{2} = 2$ is the product of irrationals, and this product is rational.

Similarly, $\frac{\sqrt{2}}{\sqrt{2}} = 1$ is the quotient of irrationals, and this quotient is rational.

To show that the product or quotient of two irrational numbers may be rational, observe that $1 + \sqrt{2}$ is irrational, as it is the sum of a rational and an irrational.

Thus, $\left(1 + \sqrt{2}\right)\left(1 + \sqrt{2}\right) = 3 + 2\sqrt{2}$ is the product of irrationals, and this product is irrational, as it is the sum of a rational and an irrational.

Also, $\frac{1+\sqrt{2}}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{\sqrt{2}} + 1$ is the quotient of irrationals and this is irrational as it is the sum of an irrational and a rational. ■

**Alternate Proof:** If we accept the fact that the square root of any integer that is not a perfect square is irrational, then we have easier proofs that the product or quotient of irrationals can be irrational:

Observe: $\sqrt{3}\sqrt{2} = \sqrt{6}$ is the product of irrationals, and this product is irrational.

Observe: $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$ is the quotient of irrationals, and this quotient is irrational. ■

16. Between any two distinct rational numbers there is another rational number.

    **Proof.** Let $x, y \in \mathbf{Q}$, with $x \neq y$.

    Without loss of generality, $x < y$.

    Define $z$ by $z = \frac{x+y}{2}$

    Observe: $x + y$ is rational, as it is the sum of rationals.

    Thus, $z = \frac{x+y}{2}$ is rational, as it is the quotient of rationals.

    Furthermore, $x = \frac{x+x}{2} < \frac{x+y}{2} < \frac{y+y}{2} = y$.

    i.e., $x < \frac{x+y}{2} < y$.

    Thus, $z = \frac{x+y}{2}$ is a rational number that lies between $x$ and $y$. ■

17. Between any two distinct real numbers there is a rational number.

    **Proof.** Let $x, y \in \mathbf{R}$, with $x \neq y$.

    Without loss of generality, $x < y$.

    Thus, $\exists\, \varepsilon > 0$ such that $y - x = \varepsilon$.

    Select a natural number $n$ such that $n\varepsilon > 1$.

    $\Rightarrow n(y - x) = n\varepsilon > 1$.

    $\Rightarrow ny - nx = n\varepsilon > 1$.

    Since $ny - nx > 1$, there exists an integer $m$ such that $nx < m < ny$.

    $\Rightarrow x < \frac{m}{n} < y$.

    Since $\frac{m}{n}$ is the quotient of integers, it is rational.

    i.e., $\frac{m}{n}$ is a rational number between $x$ and $y$. ■

**Remark 1** *The preceding proof can be extended to show that in each interval* $\left(x, \frac{m}{n}\right)$ *and* $\left(\frac{m}{n}, y\right)$ *there is a rational number (let's say* $\frac{m_x}{n_x} \in \left(x, \frac{m}{n}\right)$*, and* $\frac{m_y}{n_y} \in \left(\frac{m}{n}, y\right)$*, then in each of the four subintervals defined by these rational numbers, there is a rational number, etc. Hence, between any two distinct real numbers* $x$ *and* $y$*, there are* ***infinitely many*** *rational numbers.*

18. Between any two distinct real numbers there is an irrational number.

**Proof.** Let $x, y \in \mathbf{R}$, with $x \neq y$.

Without loss of generality, $x < y$.

$\Rightarrow \sqrt{2}x < \sqrt{2}y$

Since there exists infinitely many rational numbers between any two distinct real numbers, $\exists\ q \in \mathbb{Q}$, with $q \neq 0$, such that $\sqrt{2}x < q < \sqrt{2}y$.

$\Rightarrow x < \frac{q}{\sqrt{2}} < y$.

**Observe:** $z = \frac{q}{\sqrt{2}} \in \mathbb{Q}^C$, as it is the quotient of a nonzero rational number and an irrational number.

Thus, $z = \frac{q}{\sqrt{2}}$ is an irrational number between $x$ and $y$. ■

**Remark 2** *The preceding proof was inspired by Taoufiq Bellamine and refined by Niall McNellis.*

19. Prove or disprove:

(a) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 5x + 3$ is onto

**Proof.** We must show that $\forall y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ such that $f(x) = y$.

Let $y \in \mathbf{R}$ be given.

Let $x \in \mathbf{R}$ be given by $x = \underline{\frac{y-3}{5}}$

Observe: $f(x) = \underline{5x+3} = \underline{5\left(\frac{y-3}{5}\right)+3} = \underline{(y-3)+3} = y$.

Thus, given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ (namely $x = \underline{\frac{y-3}{5}}$) such that $f(x) = y$.

Hence, $f(x)$ is onto. ■

**Scratchwork:**

| |
|---|
| We want: $x$ such that $f(x) = y$. |
| $\Rightarrow 5x + 3 = y$ |
| $\Rightarrow 5x = y - 3$ |
| $\Rightarrow x = \frac{y-3}{5}$ |

(b) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 5x + 3$ is one to one

**Proof.** Suppose that

$$\begin{aligned} & f(x_1) &=& f(x_2) \\ \Rightarrow\ & 5x_1 + 3 &=& 5x_2 + 3 \\ \Rightarrow\ & 5x_1 &=& 5x_2 \\ \Rightarrow\ & x_1 &=& x_2 \end{aligned}$$

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Hence, $f$ is one to one. ■

(c) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 5x^2 + 3$ is onto

Note that since $x^2 \geq 0$, it follows that $5x^2 + 3 \geq 3$.

Therefore our claim is false.

To cite a specific counter-example, consider $y = 0$. There does not exist an $x \in \mathbf{R}$ such that $f(x) = 0$. ■

(d) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 5x^2 + 3$ is one to one.

This is false. As a counter-example, consider $x_1 = -1$ and $x_2 = 1$.

Observe that $x_1$ and $x_2$ are are distinct values of $x$ such that $f(x_1) = 8 = f(x_2)$.

i.e., $x_1 \neq x_2$, and yet $f(x_1) = f(x_2)$.

Hence, $f$ is not one to one. ■

(e) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 5x^3 + 3$ is onto

**Proof.** We must show that $\forall y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ such that $f(x) = y$.

Let $y \in \mathbf{R}$ be given.

Let $x \in \mathbf{R}$ be given by $x = \underline{\left(\frac{y-3}{5}\right)^{\frac{1}{3}}}$

Observe: $f(x) = \underline{5x^3 + 3} = \underline{5\left(\left(\frac{y-3}{5}\right)^{\frac{1}{3}}\right)^3 + 3} = \underline{5\left(\frac{y-3}{5}\right) + 3} = \underline{(y-3) + 3} = y$.

Thus, given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ (namely $x = \underline{\left(\frac{y-3}{5}\right)^{\frac{1}{3}}}$) such that $f(x) = y$.

Hence, $f(x)$ is onto. ■

**Scratchwork:**

| |
|---|
| We want: $x$ such that $f(x) = y$. |
| $\Rightarrow 5x^3 + 3 = y$ |
| $\Rightarrow 5x^3 = y - 3$ |
| $\Rightarrow x^3 = \frac{y-3}{5}$ |
| $\Rightarrow x = \left(\frac{y-3}{5}\right)^{\frac{1}{3}}$ |

(f) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 5x^3 + 3$ is one to one

**Proof.** Suppose that

$$\begin{array}{rrcl} & f(x_1) & = & f(x_2) \\ \Rightarrow & 5x_1^3 + 3 & = & 5x_2^3 + 3 \\ \Rightarrow & 5x_1^3 & = & 5x_2^3 \\ \Rightarrow & x_1^3 & = & x_2^3 \\ \Rightarrow & x_1 & = & x_2 \end{array}$$

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Hence, $f$ is one to one. ■

20. Prove or disprove:

(a) Prove or Disprove: $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 4x - 7$ is one-to-one.

**Proof.** Suppose that

$$\begin{array}{rrcl} & f(x_1) & = & f(x_2) \\ \Rightarrow & 4x_1 - 7 & = & 4x_2 - 7 \\ \Rightarrow & 4x_1 & = & 4x_2 \\ \Rightarrow & x_1 & = & x_2 \end{array}$$

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Hence, $f$ is one to one. ■

(b) Prove or Disprove: $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 4x - 7$ is onto

**Proof.** We must show that $\forall y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ such that $f(x) = y$.

Let $y \in \mathbf{R}$ be given.

Let $x \in \mathbf{R}$ be given by $x = \frac{y+7}{4}$

Observe: $f(x) = 4x - 7 = 4\left(\frac{y+7}{4}\right) - 7 = (y+7) - 7 = y$.

Thus, given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ (namely $x = \frac{y+7}{4}$) such that $f(x) = y$.

Hence, $f(x)$ is onto. ■

**Scratchwork:**

| |
|---|
| We want: $x$ such that $f(x) = y$.<br>$\Rightarrow 4x - 7 = y$<br>$\Rightarrow 4x = y + 7$ |
| $\Rightarrow x = \frac{y+7}{4}$ |

(c) Prove or Disprove: $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 2x^2 + 4$ is one-to-one.

**Proof.** This is false.

As a counter-example, consider:

$x_1 = -1$ and $x_2 = 1$ are distinct values of $x$ such that $f(x_1) = 6 = f(x_2)$.

i.e., $\exists\, x_1 \neq x_2$ such that $f(x_1) = f(x_2)$.

Hence, $f$ is not one to one. ■

(d) Prove or Disprove: $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 2x^2 + 4$ is onto

**Proof.** The claim is false.

Note that since $x^2 \geq 0$, it follows that $2x^2 + 4 \geq 4$.

To cite a specific counter-example, consider $y = 0$. By the aforementioned reasoning, there does not exist an $x$ such that $f(x) = y$.

Hence, $f$ is not onto. ■

(e) Prove or Disprove: $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 3x^3 + 2$ is one-to-one.

**Proof.**

$$\begin{array}{rrcl} \text{Suppose that} & f(x_1) & = & f(x_2) \\ \Rightarrow & 3x_1^3 + 2 & = & 3x_2^3 + 2 \\ \Rightarrow & 3x_1^3 & = & 3x_2^3 \\ \Rightarrow & x_1^3 & = & x_2^3 \\ \Rightarrow & x_1 & = & x_2 \end{array}$$

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Hence, $f$ is one to one. ■

(f) Prove or Disprove: $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 3x^3 + 2$ is onto

**Proof.** We must show that $\forall y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ such that $f(x) = y$.

Let $y \in \mathbf{R}$ be given.

Let $x \in \mathbf{R}$ be given by $x = \underline{\left(\frac{y-2}{3}\right)^{\frac{1}{3}}}$

Observe: $f(x) = \underline{3x^3 + 2} = \underline{3\left(\left(\frac{y-2}{3}\right)^{\frac{1}{3}}\right)^3 + 2} = \underline{3\left(\frac{y-2}{3}\right) + 2} = \underline{(y-2) + 2} = y.$

Thus, given $y \in \mathbf{R}$, $\exists x \in \mathbf{R}$ (namely $x = \underline{\left(\frac{y-2}{3}\right)^{\frac{1}{3}}}$ ) such that $f(x) = y$.

Hence, $f(x)$ is onto. ■

**Scratchwork:**

| |
|---|
| We want $x$ such that $f(x) = y$.<br>$\Rightarrow 3x^3 + 2 = y$<br>$\Rightarrow 3x^3 = y - 2$<br>$\Rightarrow x^3 = \frac{y-2}{3}$ |
| $\Rightarrow x = \left(\frac{y-2}{3}\right)^{\frac{1}{3}}$ |

21. The composition of one to one functions is one to one.

(i.e., If $f : \mathbf{X} \to \mathbf{Y}$ is one to one, and $g : \mathbf{Y} \to \mathbf{Z}$ is one to one, then $g \circ f : \mathbf{X} \to \mathbf{Z}$ is one to one.)

**Proof.** Let the hypothesis be given. (i.e., Suppose that $f : \mathbf{X} \to \mathbf{Y}$ is one to one, and $g : \mathbf{Y} \to \mathbf{Z}$ is one to one.)

Suppose also that $g(f(x_1)) = g(f(x_2))$.

$\Rightarrow f(x_1) = f(x_2)$, since $g$ is one to one.

$\Rightarrow x_1 = x_2$, since $f$ is one to one.

i.e., $g(f(x_1)) = g(f(x_2)) \Rightarrow x_1 = x_2$.

Hence, $g \circ f$ is one to one. ■

22. The composition of onto functions is onto.

(i.e., If $f : \mathbf{X} \to \mathbf{Y}$ is onto, and $g : \mathbf{Y} \to \mathbf{Z}$ is onto, then $g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto.)

**Proof.** Let the hypothesis be given. (i.e., Suppose that $f : \mathbf{X} \to \mathbf{Y}$ is onto, and $g : \mathbf{Y} \to \mathbf{Z}$ is onto.)

Let $z \in \mathbf{Z}$.

Since $g$ is onto, $\exists y \in \mathbf{Y}$ (call it $y_z$) such that $g(y_z) = z$.

Since $f$ is onto, $\exists x \in \mathbf{X}$ (call it $x_z$) such that $f(x_z) = y_z$.

**Observe:** $g(f(x_z)) = g(y_z) = z$.

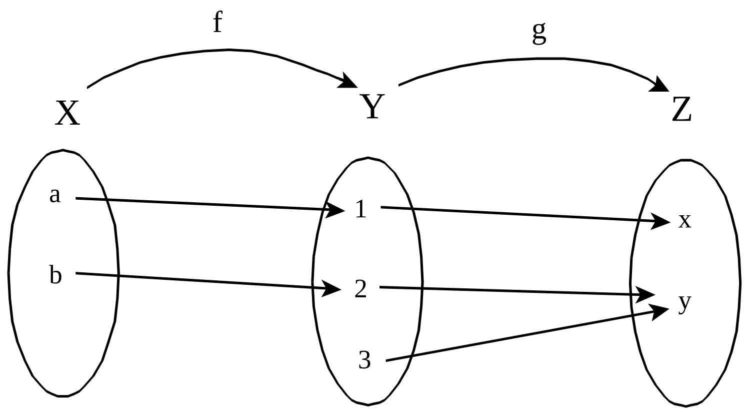
Thus, given $z \in \mathbf{Z}$, $\exists x \in \mathbf{X}$ (namely $x_z$) such that $(g \circ f)(x) = z$.

Hence, $g \circ f$ is onto. ■

23. Given $f : \mathbf{X} \to \mathbf{Y}$ and $g : \mathbf{Y} \to \mathbf{Z}$, Suppose that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is one to one. Is either $f$ or $g$ necessarily one to one?

**Claim:** $g$ is not necessarily one to one.

**Proof.** Consider $f : \mathbf{X} \to \mathbf{Y}$ and $g$: $\mathbf{Y} \to \mathbf{Z}$ as shown below. Note that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is one to one, as $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, but $g : \mathbf{Y} \to \mathbf{Z}$ is not one to one.




■

**Claim:** $f$ must be one to one.

**Proof.** Let the hypothesis be given. (i.e., suppose that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is one to one.)

Suppose also, for the sake of deriving a contradiction, that $f : \mathbf{X} \to \mathbf{Y}$ is not one to one. Then $\exists x_1$, $x_2 \in \mathbf{X}$, with $x_1 \neq x_2$, such that $f(x_1) = f(x_2)$.
$\Rightarrow g(f(x_1)) = g(f(x_2))$.

Thus, $\exists x_1,\ x_2 \in \mathbf{X}$, with $x_1 \neq x_2$, such that $g(f(x_1)) = g(f(x_2))$.

$\Rightarrow g \circ f : \mathbf{X} \to \mathbf{Z}$ is not one to one, contrary to our hypothesis.

Since the assumption that $f$ is not one to one yields a contradiction, it must be false.

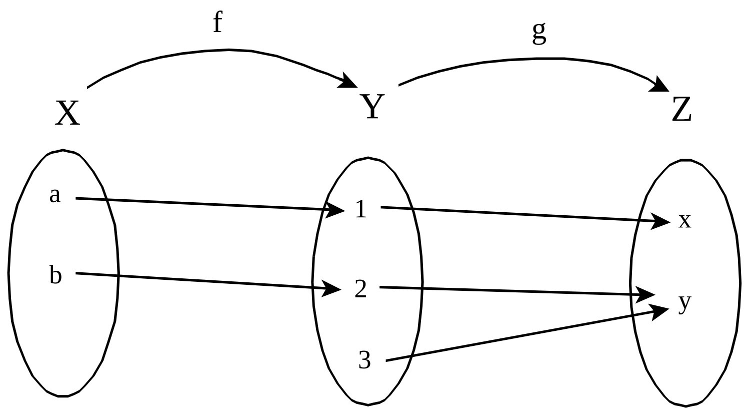
Hence, $f : \mathbf{X} \to \mathbf{Y}$ is one to one.

Thus, if $g \circ f : \mathbf{X} \to \mathbf{Z}$ is one to one, then $f : \mathbf{X} \to \mathbf{Y}$ must be onto to one , but $g : \mathbf{Y} \to \mathbf{Z}$ is not necessarily one to one. ■

24. Given $f : \mathbf{X} \to \mathbf{Y}$ and $g : \mathbf{Y} \to \mathbf{Z}$, Suppose that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto. Is either $f$ or $g$ necessarily onto?

**Claim:** $f$ is not necessarily onto.

**Proof.** Consider $f : \mathbf{X} \to \mathbf{Y}$ and $g$: $\mathbf{Y} \to \mathbf{Z}$ as shown below. Note that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto, as $\forall z \in \mathbf{Z}$,$\exists x \in \mathbf{X}$ such that $g(f(x)) = z$, and yet $f : \mathbf{X} \to \mathbf{Y}$ is not onto.




■

**Claim:** $g : \mathbf{Y} \to \mathbf{Z}$ must be onto.

**Proof.** Let the hypothesis be given. (i.e., suppose that $g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto.)Let $z \in \mathbf{Z}$ be given.

Then $\exists x \in \mathbf{X}$ such that $g(f(x)) = z$.

$\Rightarrow \exists y \in \mathbf{Y}$ (namely $y = f(x)$), such that $g(y) = z$.

i.e., Given $z \in \mathbf{Z}$, $\exists y \in \mathbf{Y}$ such that $g(y) = z$.

Hence, $g : \mathbf{Y} \to \mathbf{Z}$ must be onto.

Thus, if $g \circ f : \mathbf{X} \to \mathbf{Z}$ is onto, then $g : \mathbf{Y} \to \mathbf{Z}$ must be onto, but $f : \mathbf{X} \to \mathbf{Y}$is not necessarily onto. ■

25. A function $f : \mathbf{X} \rightarrow \mathbf{Y}$ has an inverse if and only if it is one to one and onto.

**Proof.** If $f : \mathbf{X} \rightarrow \mathbf{Y}$ has an inverse then it is one to one and onto.

Let the hypothesis be given. (i.e., suppose that $f : \mathbf{X} \rightarrow \mathbf{Y}$ has an inverse, $f^{-1}$)

Then: $f^{-1} \circ f = 1_X$, and hence, $f^{-1} \circ f$ is one to one and onto.

Also: $f \circ f^{-1} = 1_Y$, and hence, $f \circ f^{-1}$is one to one and onto.

Since $f^{-1} \circ f$ is one to one and onto, then by previous exercises (23 and24), $f$ must be one to one, and $f^{-1}$ must be onto.

Since $f \circ f^{-1}$ is one to one and onto, then by previous exercises (23 and24), $f^{-1}$ must be one to one, and $f$ must be onto.

Hence, both $f$ and $f^{-1}$ are one to one and onto.

If $f : \mathbf{X} \rightarrow \mathbf{Y}$ is one to one and onto, then it has an inverse.

Let the hypothesis be given. (i.e., Suppose that $f : \mathbf{X} \rightarrow \mathbf{Y}$ is one to one and onto.)
Note that since $f$ is onto, for any value of $y \in \mathbf{Y}$, there exists an $x \in \mathbf{X}$ such that $f(x) = y$.

Since $f$ is one to one, there is *only* one $x \in \mathbf{X}$ such that $f(x) = y$. We'll call it $x_y$

Thus, we can define $f^{-1} : \mathbf{Y} \rightarrow \mathbf{X}$ by $f^{-1}(y) = x_y$

We must now check and make sure that $f^{-1} \circ f = 1_X$ and that $f \circ f^{-1} = 1_Y$.

**Observe:** $f^{-1} \circ f(x_y) = f^{-1}(f(x_y)) = f^{-1}(y) = x_y$.

Thus, $f^{-1} \circ f = 1_X$

**Also:** $f \circ f^{-1}(y) = f\left(f^{-1}(y)\right) = f(x_y) = y$

Hence, $f$ has an inverse. ■

**Remark 3** *Note that if* $\exists f : \mathbf{X} \rightarrow \mathbf{Y}$ *that is one to one and onto, then* $\exists g : \mathbf{Y} \rightarrow \mathbf{X}$ *that is one to one and onto (e.g.,* $f^{-1}$*)*

**Remark 4** *If* $\exists f : S \rightarrow \mathbf{N}$ *that is one to one and onto, then* $\exists g : \mathbf{N} \rightarrow \mathbf{S}$ *that is one to one and onto also.*

*Thus, to show that a set is denumerable, we can show that* $\exists g : \mathbf{N} \rightarrow \mathbf{S}$ *that is one to one and onto, or we can show that* $\exists f : S \rightarrow \mathbf{N}$ *that is one to one and onto. Either is sufficient.*

**Remark 5** *Since the composition of one to one and onto functions is also one to one and onto, if a set A is known to be denumerable, then any set B that can be put into a one to one correspondence with A is also*

*one to one and onto. (Since A is denumerable,* $\exists\ f : \mathbf{N} \overset{one\ to\ one}{\underset{onto}{\longrightarrow}} A$.

*Similarly,* $\exists\ g : \mathbf{A} \overset{one\ to\ one}{\underset{onto}{\longrightarrow}} B$. *Thus,* $(g \circ f) : \mathbf{N} \rightarrow B$ *is one to one and onto.)*

**The point is this:** An alternate way of showing that a set $B$ is denumerable is to exhibit a one to one correspondence between $B$ and a set $A$, where $A$ is known to be denumerable.

26. The set of even natural numbers $\mathbf{E} = \{2, 4, 6, 8, \ldots\}$ is denumerable.

**Proof.** Observe:

$$\begin{array}{rcccccccccl} \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & \ldots & \} \\ & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \mathbf{E} & = & \{ & 2, & 4, & 6, & 8, & 10, & 12, & \ldots & \} \end{array}$$

Define $f : \mathbf{N} \to \mathbf{E}$ by $f(n) = 2n$

Clearly from the diagram above, $f$ is one to one and onto. Hence, $\mathbf{E}$ is denumerable. ■

27. The set of odd natural numbers $\mathbf{O} = \{1, 3, 5, 7, \ldots\}$ is denumerable.

**Proof.** Observe:

$$\begin{array}{rcccccccccl} \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & \ldots & \} \\ & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \mathbf{O} & = & \{ & 1, & 3, & 5, & 7, & 9, & 11, & \ldots & \} \end{array}$$

Define $f : \mathbf{N} \to \mathbf{O}$ by $f(n) = 2n - 1$.

Clearly from the diagram above, $f$ is one to one and onto. Hence, $\mathbf{O}$ is denumerable. ■

28. The set of integers $\mathbf{Z} = \{0, \pm 1, \pm 2 \pm 3, \pm 4, \ldots\}$ is denumerable.

**Proof.** Observe:

$$\begin{array}{rccccccccccl} \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots & \} \\ f\downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \mathbf{Z} & = & \{ & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \ldots & \} \end{array}$$

Define $f : \mathbf{N} \to \mathbf{Z}$ by $f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$

Clearly from the diagram above, $f$ is one to one and onto. Hence, $\mathbf{Z}$ is denumerable. ■

29. The set $5\mathbf{Z} = \{0, \pm 5, \pm 10 \pm 15, \pm 20, \ldots\}$ is denumerable.

**Proof.** Observe:

$$\begin{array}{rccccccccccl} \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots & \} \\ f\downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ 5\mathbf{Z} & = & \{ & 0, & 5, & -5, & 10, & -10, & 15, & -15, & \ldots & \} \end{array}$$

Define $f : \mathbf{N} \to \mathbf{5Z}$ by $f(n) = \begin{cases} \frac{5n}{2} & \text{if } n \text{ is even} \\ -\frac{5(n-1)}{2} & \text{if } n \text{ is odd} \end{cases}$

Clearly from the diagram above, $f$ is one to one and onto. Hence, $5\mathbf{Z}$ is denumerable. ■

30. $\mathbf{Z} \sim \mathbf{nZ}$ (and consequently, $\mathbf{nZ}$ is denumerable)

**Proof.** Observe:

$$\begin{array}{rccccccccccl} \mathbf{Z} & = & \{ & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \ldots & \} \\ & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ n\mathbf{Z} & = & \{ & 0, & n, & -n, & 2n, & -2n, & 3n, & -3n, & \ldots & \} \end{array}$$

Define $f : \mathbf{Z} \to \mathbf{nZ}$ by $f(k) = kn$

Clearly from the diagram above, $f$ is one to one and onto. Hence, $\mathbf{Z} \sim n\mathbf{Z}$, and therefore, $n\mathbf{Z}$ is denumerable by Remark 5 on page 14. ■

31. The set of positive rational numbers $\mathbf{Q}^+$ is denumerable.

    Consider the table of ordered pairs below:

$$\begin{array}{ccccccccccc}
(1,1) & \rightarrow & (1,2) & & (1,3) & \rightarrow & (1,4) & & (1,5) & \rightarrow & \ldots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & & \\
(2,1) & & (2,2) & & (2,3) & & (2,4) & & (2,5) & \ldots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & & \\
(3,1) & & (3,2) & & (3,3) & & (3,4) & & (3,5) & \ldots & \\
 & \swarrow & & \nearrow & & & & & & & \\
(4,1) & & (4,2) & & (4,3) & & (4,4) & & (4,5) & \ldots & \\
\downarrow & \nearrow & & & & & & & & & \\
(5,1) & & (5,2) & & (5,3) & & (5,4) & & (5,5) & \ldots & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

    If we consider the ordered pair $(i,j)$ in the i$^{th}$ row and j$^{th}$ column to represent the quotient of integers $\frac{i}{j}$, then every positive rational number appears in the table at least once. (e.g., the positive rational number $\frac{m}{n}$ appears in the m$^{th}$ row and $n^{th}$ column.)

    Furthermore, the arrows in the table induce an **exhaustive ordering** of the positive rational numbers as follows:

    $1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, 5, \frac{1}{5}, \ldots$

    (Note that we have discarded repetitions of rationals if they occur. e.g., we have discarded $(2,2)$ because it is equivalent to $(1,1)$ which is already on our list.)

    Note also that since the positive rationals are **ordered,** they are in a one to one correspondence with the natural numbers.

    Hence, the positive rational numbers are denumerable. ■

32. The set of negative rational numbers $\mathbf{Q}^-$ is denumerable.

    **Proof.** The function $f : \mathbf{Q}^+ \to \mathbf{Q}^-$ given by $f\left(\frac{m}{n}\right) = -\frac{m}{n}$ is clearly one to one and onto.

    For if $f(x_1) = f(x_2)$,

    Then $-x_1 = -x_2$

    $\Rightarrow x_1 = x_2,$ thus $f$ is one to one.

    Also, given $y \in \mathbf{Q}^-$, we can choose $x \in \mathbf{Q}^+$, given by $x = -y$.

    This yields $f(x) = -x = -(-y) = y$.

    Thus, $f$ is onto. ■

33. The union of a denumerable set and a finite set is denumerable (you can assume that the two sets are disjoint).

**Proof.** Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, b_3, \ldots\}$.

Then $A$ is finite and $B$ is denumerable.

Observe:

$$\begin{array}{ccccccccccccc} \mathbf{N} & = & \{ & 1, & 2, & 3, & \ldots, & k, & k+1, & k+2, & k+3, & \ldots & \} \\ f\downarrow & & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \\ (A \cup B) & = & \{ & a_1, & a_2, & a_3, & \ldots, & a_k & b_1, & b_2, & b_3, & \ldots & \} \end{array}$$

Define $f : \mathbf{N} \to (A \cup B)$ by $f(n) = \begin{cases} a_n & \text{if } n \leq k \\ b_{n-k} & \text{if } n > k \end{cases}$

Clearly from the diagram above, $f$ is one to one and onto. Hence, $(A \cup B)$ is denumerable. ■

34. The union of two (disjoint) denumerable sets is denumerable.

**Proof.** Let $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$

Observe:

$$\begin{array}{cccccccccccc} \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & \ldots & \} \\ f\downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ (A \cup B) & = & \{ & a_1, & b_1, & a_2, & b_2, & a_3, & b_3, & \ldots & \} \end{array}$$

Define $f : \mathbf{N} \to (A \cup B)$ by $f(n) = \begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ b_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$

Clearly from the diagram above, $f$ is one to one and onto. Hence, $(A \cup B)$ is denumerable. ■

35. The union of finitely many (disjoint) denumerable sets is denumerable (i.e., if $A_1, A_2, \ldots, A_n$ are denumerable, then $\cup_{i=1}^{n} A_i$ is denumerable.)

**Proof.** Suppose that $A_1, A_2, \ldots, A_n$ are denumerable. Then we can name their elements as follows:

$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$

$\vdots$

$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$

Consider:

$$\begin{array}{cccccccccccccccc} \mathbf{N} & = & \{ & 1, & 2, & & n, & n+1, & n+2, & & 2n, & 2n+1, & 2n+2, & & 3n, & \ldots \\ f\uparrow & & & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow & \\ \cup_{i=1}^{n} A_i & = & \{ & a_{11}, & a_{21}, & \ldots, & a_{n1}, & a_{12}, & a_{22} & \ldots, & a_{n2}, & a_{13}, & a_{23}, & \ldots, & a_{n3}, & \ldots \end{array}$$

The function $f : \cup_{i=1}^{n} A_i \to \mathbf{N}$ given by $f(a_{ij}) = (j-1)n + i$ as shown above, is clearly one to one and onto. Hence, $\cup_{i=1}^{n} A_i$ is denumerable. ■

**Alternatively:** Consider the function $f : \mathbf{N} \to \cup_{i=1}^{n} A_i$

$$\begin{array}{ccccccccccccccc} \mathbf{N} & = & \{ & 1, & 2, & & n, & n+1, & n+2, & & 2n, & 2n+1, & 2n+2, & & 3n, & \ldots \\ f\downarrow & & & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \\ \cup_{i=1}^{n} A_i & = & \{ & a_{11}, & a_{21}, & \ldots, & a_{n1}, & a_{12}, & a_{22} & \ldots, & a_{n2}, & a_{13}, & a_{23}, & \ldots, & a_{n3}, & \ldots \end{array}$$

The function $f : \mathbf{N} \to \cup_{i=1}^{n} A_i$ as shown above, is clearly one to one and onto. Hence, $\cup_{i=1}^{n} A_i$ is denumerable. ■

36. **Alternate Proof**

(By induction on $n$.)

Suppose that $A_1, A_2, \ldots, A_n, \ldots$ are denumerable.

(Step 1) Show that our proposition is true for $n = 1$

$\cup_{i=1}^{1} A_i = A_1$, which is denumerable, by hypothesis.

(Step 2) Assume that $\cup_{i=1}^{k} A_i$ is denumerable, and show that $\cup_{i=1}^{k+1} A_i$ is denumerable.

**Observe:** $\cup_{i=1}^{k+1} A_i = \underbrace{\left(\cup_{i=1}^{k} A_i\right)}_{\text{denumerable by ind. hypothesis}} \cup A_{k+1}$, which is denumerable, since it is the union of two denumerable sets.

Hence, $\cup_{i=1}^{n} A_i$ is denumerable for all $n \in \mathbf{N}$. ■

37. The union of denumerably many denumerable sets is denumerable (i.e., if $A_1, A_2, \ldots, A_n, \ldots$ are denumerable, then $\cup_{i=1}^{\infty} A_i$ is countable.) (Again, you can assume that the sets are disjoint.)

    **Proof.** Let sets $A_1, A_2, \ldots, A_n, \ldots$ be denumerable and given by:

    $A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$

    $A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$

    $A_3 = \{a_{31}, a_{32}, a_{33}, \ldots\}$

    $\vdots$

    $A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$

    $\vdots$

    (Note that $a_{ij}$ is the $j^{th}$ element of the $i^{th}$ set.)

    Consider the table of elements from $\cup_{i=1}^{\infty} A_i$ listed below:

$$
\begin{array}{ccccccccccc}
a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} & & a_{15} & \rightarrow & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & & \\
a_{21} & & a_{22} & & a_{23} & & a_{24} & & a_{25} & \cdots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & & \\
a_{31} & & a_{32} & & a_{33} & & a_{34} & & a_{35} & \cdots & \\
 & \swarrow & & \nearrow & & & & & & & \\
a_{41} & & a_{42} & & a_{43} & & a_{44} & & a_{45} & \cdots & \\
\downarrow & \nearrow & & & & & & & & & \\
a_{51} & & a_{52} & & a_{53} & & a_{54} & & a_{55} & \cdots & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

    The table contains every element of $\cup_{i=1}^{\infty} A_i$. For example, the $j^{th}$ element of set $A_i$ is given by $a_{ij}$. This element is found in the $i^{th}$ row and $j^{th}$ column of the table.

    Furthermore, the arrows in the table induce an exhaustive **ordering** of the elements of $\cup_{i=1}^{\infty} A_i$ as follows:

    $a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, a_{23}, a_{32}, a_{41}, a_{51}, a_{42}, a_{33}, a_{24}, a_{15}, \ldots$

    Note also that since the entire set of elements of $\cup_{i=1}^{\infty} A_i$ is **ordered,** they are in a one to one correspondence with the natural numbers.

    Hence, the union of denumerably many denumerable sets is denumerable. ■

38. The set of rational numbers is denumerable (countable).

    **Proof.** $\mathbf{Q}^+ \cup \{0\}$ is the union of a denumerable set and a finite set, hence it is denumerable.

    The entire set of rationals can be expressed as $\mathbf{Q} = (\mathbf{Q}^+ \cup \{0\}) \cup \mathbf{Q}^-$, which is the union of two denumerable sets, hence denumerable. ■

39. The real numbers 0.5 and $0.499999\ldots$ are equal. (i.e., $0.5 = 0.4999\ldots$)

Suppose that $x = 0.4999\ldots$

**Observe:** $10x = 4.999\ldots$

Hence: $9x = 10x - x = (4.999\ldots) - (0.4999\ldots) = 4.5$

i.e., $9x = 4.5$

Hence, $x = 0.5$

But $x = 0.4999\ldots$ also.

Hence $0.5 = 0.4999\ldots$ ■

**Remark:** The previous proof hinges upon the supposition that we know how to add and subtract non-terminating decimals and that when we do, "things work out" just as we think they should.

40. **Alternate Proof**

Suppose, for the sake of deriving a contradiction, that $0.5 \neq 0.4999\ldots$

Then $0.5 > 0.4999\ldots$ and consequently, $\exists \varepsilon > 0$ such that $0.5 - 0.4999\ldots = \varepsilon$

By the Axiom of Archimedes, $\exists n \in \mathbf{N}$ such that $n > -\log(\varepsilon)$

**Observe:** Since $0.4999\ldots > \underbrace{0.499\ldots9}_{n \text{ decimal places}}$,

It follows that $\varepsilon = 0.5 - 0.4999\ldots < 0.5 - \underbrace{0.499\ldots9}_{n \text{ decimal places}} = 10^{-n} < 10^{\log(\varepsilon)} = \varepsilon$

Thus, we have: $\varepsilon < \varepsilon$, a contradiction.

Since the assumption that $0.5 \neq 0.4999\ldots$ leads to a contradiction, the assumption must be false. Hence, $0.5 = 0.4999\ldots$ ■

41. The set of real numbers in the interval $[0,1]$ is uncountable (non-denumerable).

**Proof.** (By contradiction)

Suppose, for the sake of deriving a contradiction, that the set of real numbers in the interval $[0,1]$ is denumerable.

Then there exists an *exhaustive ordering* of the set of real numbers in the interval $[0,1]$.

$$\{x_1, x_2, x_3, \ldots, x_n, \ldots\}$$

Note that this ordering contains ALL of the real numbers in the interval $[0,1]$.

Consider the decimal expansions of these numbers:

$x_1 = 0.x_{11}x_{12}x_{13}\ldots$

$x_2 = 0.x_{21}x_{22}x_{23}\ldots$

$x_3 = 0.x_{31}x_{32}x_{33}\ldots$

$\vdots$

$x_n = 0.x_{n1}x_{n2}x_{n3}\ldots x_{nn}\cdots$

$\vdots$

**Observe:** Here, $x_{ij}$ is the j$^{th}$ digit past the decimal point in the decimal expansion of the i$^{th}$ real number $x_i$.

**Also:** If $x_i$ can be written in terminating and non-terminating form (e.g., 0.5 can be written as 0.499999...), then we choose the non-terminating form.

(The number 0 will be represented as 0.000...)

Define $y \in [0,1]$ as follows:

$y = 0.y_1y_2y_3\ldots y_{n\ldots}$ where $y_i$ is the i$^{th}$ digit past the decimal point in the decimal expansion of $y$.

For $n = 1, 2, 3, \ldots$ define the digit $y_n$ as follows:

$$y_n = \begin{cases} 5 & \text{if } x_{nn} \neq 5 \\ 1 & \text{if } x_{nn} = 5 \end{cases}$$

Observe: $y \in [0,1]$ and yet $y \neq x_n$ for any $n \in \mathbf{N}$.

The reason for this is that, by construction of $y$, the $n^{th}$ digit of $y$ is different from the $n^{th}$ digit of $x_n$ (i.e., $y_n \neq x_{nn}$) for all $n \in \mathbf{N}$.

Hence, $y \neq x_n \ \forall n \in \mathbf{N}$.

This contradicts our assumption that our list contains ALL of the real numbers in the interval $[0,1]$.

Since the assumption that the set of real numbers in the interval $[0,1]$ is denumerable led to this contradiction, the assumption must be false. Hence, the numbers in the interval $[0,1]$ is non-denumerable (uncountable). ■