Problem List For Test #2 - Solutions

Fall 2018

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1. **Prove:** $0 \leq |x| \leq r$ if and only if $-r \leq x \leq r$

Proof. $\boxed{0 \leq |x| \leq r \Rightarrow -r \leq x \leq r}$

Suppose that $0 \leq |x| \leq r$.

Case 1: If $x \geq 0$, then $|x| = x,$ and consequently, $0 \leq x \leq r$, by our hypothesis.

$\Rightarrow -r \leq 0 \leq x \leq r \Rightarrow -r \leq x \leq r$

Case 2: If $x < 0$, then $|x| = -x,$ and consequently, $0 \leq -x \leq r$, by our hypothesis.

$\Rightarrow -r \leq x \leq 0 \Rightarrow -r \leq x \leq 0 \leq r \Rightarrow -r \leq x \leq r$

In both cases, $-r \leq x \leq r$.

$\boxed{-r \leq x \leq r \Rightarrow 0 \leq |x| \leq r}$

Suppose that $-r \leq x \leq r$

Case 1: If $x \geq 0$, then $0 \leq x \leq r$ by our hypothesis.

Also, if $x \geq 0$, then $|x| = x,$ and consequently, $0 \leq |x| \leq r$.

Case 2: If $x < 0$, then $-r \leq x < 0$, by our hypothesis.

Also, if $x < 0$, then $|x| = -x,$ or equivalently, $-|x| = x$.

Consequently the equation $-r \leq x < 0$ becomes $-r \leq -|x| < 0$.

Multiplying both sides by -1, we have:

$r \geq |x| > 0$

$\Rightarrow 0 \leq |x| \leq r$

In both cases, $0 \leq |x| \leq r$. ■

2. **Prove:** For all real numbers r, $0 < r < |x|$ if and only if $x < -r$ or $r < x$

Proof. We must prove that:

i. $0 < r < |x| \Rightarrow x < -r$ or $r < x$

and

ii. $x < -r$ or $r < x \Rightarrow 0 < r < |x|$

$0 < r < \lvert x \rvert \Rightarrow x < -r$ or $r < x$

The contrapositive of this statement is:

$-r \leq x$ and $x \leq r \Rightarrow 0 \leq |x| \leq r$

i.e., $-r \leq x \leq r \Rightarrow 0 \leq |x| \leq r$

But this is true by the previous theorem.

Hence, $0 < r < |x| \Rightarrow x < -r$ or $r < x$.

$x < -r$ or $r < x \Rightarrow 0 < r < \lvert x \rvert$

The contrapositive of this statement is:

$0 \leq |x| \leq r \Rightarrow -r \leq x$ and $x \leq r$

i.e., $0 \leq |x| \leq r \Rightarrow -r \leq x \leq r$

But this is true by the previous theorem.

Hence, $x < -r$ or $r < x \Rightarrow 0 < r < |x|$.

Thus, for all real numbers r, $0 < r < |x|$ if and only if $x < -r$ or $r < x$. ■

3. **Prove:** Triangle Inequality (Part #1)

   $|x+y| \leq |x| + |y|$

   Proof. Observe: $\forall x, y \in \mathbf{R}$, $2xy \leq 2|x||y|$

   $\Rightarrow x^2 + 2xy + y^2 \leq x^2 + 2|x||y| + y^2$

   $\Rightarrow x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2$ (because $x^2 = |x|^2$)

   $\Rightarrow (x+y)^2 \leq (|x| + |y|)^2$

   $\Rightarrow \sqrt{(x+y)^2} \leq \sqrt{(|x|+|y|)^2}$

   $\Rightarrow |x+y| \leq |x| + |y|$ (because $\sqrt{x^2} = |x|$) ■

4. **Prove:** Triangle Inequality (Part #1a)

   $|x-y| \leq |x| + |y|$

   Proof. By the Triangle Inequality Part #1, $|a+b| \leq |a| + |b|$

   Let $x = a$ and let $y = -b$ (i.e., $b = -y$)
   Then $|a+b| \leq |a| + |b|$ becomes $|x + (-y)| \leq |x| + |-y| \Rightarrow |x-y| \leq |x| + |y|$ ■

5. **Prove:** Triangle Inequality (Part #2)

   $|x| - |y| \leq |x+y|$

   Proof. By the Triangle Inequality (Part #1), $|a+b| \leq |a| + |b|$

   If we let $x = a + b$ and $y = -b$ (and consequently $x + y = a$), then this becomes:

   $|x| \leq |x+y| + |-y|$

   $\Rightarrow |x| \leq |x+y| + |y|$

   $\Rightarrow |x| - |y| \leq |x+y|$ ■

Remark 1 *Our motivation for this proof was as follows:*

We wanted to derive Triangle Inequality (Part #2) from (Part #1).

So, we take Part #2 and rearrange it to look like Part #1, as follows:

$|x| - |y| \leq |x + y|$

$\Leftrightarrow |x| \leq |x + y| + |y|$.

Then we pair up the variables from the preceding inequality with the standard form of the Triangle Inequality Part #1:

$$\begin{array}{ccccc} |x| & \leq & |x+y| & + & |y| \\ \updownarrow & & \updownarrow & & \updownarrow \\ |a+b| & \leq & |a| & + & |b| \end{array}$$

Thus, we have:

$$\begin{array}{ccccccc} x & & & = & a & + & b \\ x & + & y & = & a & & \\ \hline & & -y & = & & & b \quad \textit{(subtract)} \end{array}$$

$\Rightarrow x = a + b$; $y = -b$ *and* $x + y = (a + b) - b = a$

Thus, $|a + b| \leq |a| + |b|$ *becomes:*

$|x| \leq |x + y| + |-y|$

$\Rightarrow |x| \leq |x + y| + |y|$

$\Rightarrow |x| - |y| \leq |x + y|$

6. Triangle Inequality (Part #2a)

   $|y| - |x| \leq |x + y|$

   Proof. From the Triangle Inequality Part #2, $|x| - |y| \leq |x + y|$

   Interchanging the roles of x and y, we have:

   $|y| - |x| \leq |y + x|$

   $\Rightarrow |y| - |x| \leq |x + y|$ ■

7. Triangle Inequality (Part #3)

   $|x| - |y| \leq |x - y|$

   Proof. From the Triangle Inequality Part #2, we have: $|a| - |b| \leq |a + b|$

   Letting $x = a$ and $y = -b$ (i.e., $b = -y$), the inequality $|a| - |b| \leq |a + b|$ becomes:

   $|x| - |-y| \leq |x + (-y)| \Rightarrow |x| - |y| \leq |x - y|$

   i.e., $|x| - |y| \leq |x - y|$ ■

8. Triangle Inequality (Part #3a)

   $|y| - |x| \leq |x - y|$

   Proof. From the Triangle Inequality Part #2, we have: $|x| - |y| \leq |x - y|$

   Interchanging the roles of x and y, it follows that $|y| - |x| \leq |y - x|$

   But: $|y - x| = |x - y|$

   Hence, $|y| - |x| \leq |y - x| = |x - y|$

   i.e., $|y| - |x| \leq |x - y|$ ■

9. Triangle Inequality (Part #4)

   $||x| - |y|| \leq |x + y|$

   Proof. From the Triangle Inequality Part #2, we have: $|x| - |y| \leq |x + y|$

   From the Triangle Inequality Part #2a, we have: $|y| - |x| \leq |x + y|$

   $\Rightarrow -(|x| - |y|) \leq |x + y|$

   i.e., $|x| - |y| \leq |x + y|$ and $-(|x| - |y|) \leq |x + y|$.

   Thus, $||x| - |y|| \leq |x + y|$ ■

10. Triangle Inequality (Part #5)

    $||x| - |y|| \leq |x - y|$

    Proof. From the Triangle Inequality Part #4, we have: $||a| - |b|| \leq |a + b|$

    Letting $x = a$ and $y = -b$ (i.e., $b = -y$), the inequality $||a| - |b|| \leq |a + b|$ becomes:

    $||x| - |-y|| \leq |x + (-y)| \Rightarrow ||x| - |y|| \leq |x - y|$

    i.e., $||x| - |y|| \leq |x - y|$ ■

11. **Prove:** $|ab| = |a|\,|b|$ for all real numbers, a and b.

Proof. Observe:

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a|\,|b| \quad \blacksquare$$

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

12. **Prove:** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for all real numbers, a and $b \neq 0$.

$$\left|\frac{a}{b}\right| = \sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|} \quad \blacksquare$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

13. **Prove:** $\lim_{x\to 2}\left(x^2-3x+5\right)=3$

Proof. Let $\varepsilon>0$ be given.

Choose $\delta=\delta(\varepsilon)=\underline{\min\left(\frac{\varepsilon}{2},1\right)}$

Let $0<|x-2|<\delta$

Observe: $|(x^2-3x+5)-3|=|x^2-3x+2|=|(x-2)(x-1)|=\underbrace{|x-2|}_{<\delta}\underbrace{|x-1|}_{<2}$

$$<\delta\cdot 2\leq\frac{\varepsilon}{2}\cdot 2=\varepsilon.$$

i.e., $|x-2|<\delta\Rightarrow|(x^2-3x+5)-3|<\varepsilon$

Hence, $\lim_{x\to 2}\left(x^2-3x+5\right)=3$ ■

Scratch Work: We want $|(x^2-3x+5)-3|<\varepsilon$ whenever $0<|x-2|<\delta$.

$\Rightarrow|(x^2-3x+5)-3|=|x^2-3x+2|=|(x-2)(x-1)|=\underbrace{|x-2|}_{<\delta}|x-1|$

We have to place an upper bound on the size of $|x-1|$

Since $x\to 2$, we can assume, without loss of generality, that $|x-2|<1$.

Note: Since $|x-2|<\delta$, the previous statement implies that $\delta<1$.

$\Rightarrow -1<x-2<1\Rightarrow 0<x-1<2$

i.e., $|x-1|<2$.

Continuing where we left off, we have:

$|(x^2-3x+5)-3|=\underbrace{|x-2|}_{<\delta}\underbrace{|x-1|}_{<2}<\delta\cdot 2=\varepsilon$

Thus we have $\delta\cdot 2=\varepsilon\Rightarrow\delta=\frac{\varepsilon}{2}$

14. **Prove:** $\lim_{x\to 1}(x^3-7)=-6$

Proof. Let $\varepsilon > 0$ be given.

Choose $\delta = \delta(\varepsilon) = \underline{\min\left(\frac{\varepsilon}{7}, 1\right)}$

Let $0 < |x-1| < \delta$

Observe: $|(x^3-7)-(-6)| = |(x^3-1)| = |(x-1)(x^2+x+1)| = \underbrace{|x-1|}_{<\delta}\underbrace{|x^2+x+1|}_{<7}$

$< \delta \cdot 7 \le \frac{\varepsilon}{7}\cdot 7 = \varepsilon$

i.e., $|x-1| < \delta \Rightarrow |(x^3-7)-(-6)| < \varepsilon$

Hence, $\lim_{x\to 1}(x^3-7)=-6$

Scratch Work: We want $|(x^3-7)-(-6)| < \varepsilon$ whenever $0 < |x-1| < \delta$.

$\Rightarrow |(x^3-7)-(-6)| = |(x^3-1)| = |(x-1)(x^2+x+1)| = \underbrace{|x-1|}_{<\delta}|x^2+x+1|$

We need to place an upper bound on the magnitude of $|x^2+x+1|$

Since $x \to 1$, we can assume, without loss of generality, that $|x-1| < 1$.

Note: Since $|x-1| < \delta$, the previous statement implies that $\delta < 1$.

$\Rightarrow -1 < x-1 < 1 \quad \Rightarrow \quad 0 < x < 2 \quad \Rightarrow \quad |x| < 2.$

Hence, $|x^2+x+1| \le |x^2| + |x| + |1| \le 4+2+1 = 7$

i.e., $|x^2+x+1| \le 7$

Continuing where we left off, we have:

$|(x^3-7)-(-6)| = \underbrace{|x-1|}_{<\delta}\underbrace{|x^2+x+1|}_{<7} < \delta\cdot 7 = \varepsilon$

i.e., $\delta \cdot 7 = \varepsilon \Rightarrow \delta = \frac{\varepsilon}{7}$

15. **Prove:** $\lim_{x\to\infty} \frac{2x}{x+1} = 2$

Remark 2 *In order to show that* $\lim_{x\to\infty} \frac{2x}{x+1} = 2$, *we must show that given* $\varepsilon > 0$, *there exists an* $\mathbf{M} = \mathbf{M}(\varepsilon) > 0$ *such that* $x > \mathbf{M} \Rightarrow \left|\frac{2x}{x+1} - 2\right| < \varepsilon$.

Remark 3 *Here,* $\mathbf{M}$ *is not necessarily an integer –* $\mathbf{M}$ *can be any real number.*

Proof. Let $\varepsilon > 0$ be given.

Choose $\mathbf{M} = \mathbf{M}(\varepsilon) = \frac{2}{\varepsilon}$.

Observe: Given $x > \mathbf{M}$, we have:

$$\left|\frac{2x}{x+1} - 2\right| = \left|\frac{2x}{x+1} - \frac{2(x+1)}{x+1}\right| = \left|\frac{2x-2x-2}{x+1}\right| = \left|\frac{-2}{x+1}\right| = \frac{2}{x+1} < \frac{2}{x} < \frac{2}{M} = \frac{2}{\left(\frac{2}{\varepsilon}\right)} = \varepsilon$$

i.e., Given $\varepsilon > 0$, there exists an $\mathbf{M} > 0$ such that $x > \mathbf{M} \Rightarrow \left|\frac{2x}{x+1} - 2\right| < \varepsilon$.

Hence, $\lim_{x\to\infty} \frac{2x}{x+1} = 2$ ■

Remark 4 *Since* $x > \mathbf{M} > 0$, *by hypothesis, it follows that* $x + 1 > 0$, *hence, we were justified in taking the step:* $\left|\frac{-2}{x+1}\right| = \frac{2}{x+1}$.

Scratch Work: Given $x > \mathbf{M}$, we want $\left|\frac{2x}{x+1} - 2\right| < \varepsilon$

$$\Rightarrow \left|\frac{2x}{x+1} - 2\right| = \left|\frac{2x}{x+1} - \frac{2(x+1)}{x+1}\right| = \left|\frac{2x-2x-2}{x+1}\right| = \left|\frac{-2}{x+1}\right| = \frac{2}{x+1} < \frac{2}{x} < \frac{2}{M} = \varepsilon$$

i.e., we want: $\frac{2}{M} = \varepsilon \Rightarrow \frac{2}{\varepsilon} = M$

So let $\mathbf{M} = \mathbf{M}(\varepsilon) = \frac{2}{\varepsilon}$

Remark: Notice that in our Scratch Work we had: $\frac{2}{x+1} < \frac{2}{x} < \frac{2}{M} \leq \varepsilon$.

Consequently, we had: $\frac{2}{M} \leq \varepsilon \Rightarrow \frac{2}{\varepsilon} \leq M$,

and we chose: $\mathbf{M} = \mathbf{M}(\varepsilon) = \frac{2}{\varepsilon}$.

It is reasonable to ask if we could have approached it this way:

$\frac{2}{x+1} < \frac{2}{M+1} = \varepsilon$.

Consequently, we have: $\frac{2}{M+1} = \varepsilon \quad \Rightarrow \quad \frac{2}{\varepsilon} = M + 1 \quad \Rightarrow \quad \frac{2}{\varepsilon} - 1 = M.$

and we chose: $\mathbf{M} = \mathbf{M}(\varepsilon) = \frac{2}{\varepsilon} - 1$.

This will work just fine. In our proof, we will have:

$$\left|\frac{2x}{x+1} - 2\right| = \ldots = \frac{2}{x+1} < \frac{2}{M+1} = \frac{2}{\left(\frac{2}{\varepsilon}-1\right)+1} = \frac{2}{\left(\frac{2}{\varepsilon}\right)} = \varepsilon$$

16. **Prove:** $\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$

Proof. We must show that given any $\mathbf{M} \in \mathbf{R}$, with $\mathbf{M} > 0$, $\exists \delta = \delta(\mathbf{M}) > 0$, such that $0 < |x-1| < \delta \Rightarrow \frac{1}{(x-1)^2} > \mathbf{M}$.

So let $\mathbf{M} > 0$ be given.

Choose $\delta = \delta(\mathbf{M}) = \underline{\frac{1}{\sqrt{\mathbf{M}}}}$

Observe: If $|x-1| < \delta$, then we have:

$$\frac{1}{(x-1)^2} = \frac{1}{\left(\sqrt{(x-1)^2}\right)^2} = \frac{1}{(|x-1|)^2} > \frac{1}{\delta^2} = \frac{1}{\left(\frac{1}{\sqrt{\mathbf{M}}}\right)^2} = \mathbf{M}$$

i.e., $|x-1| < \delta \Rightarrow \frac{1}{(x-1)^2} > \mathbf{M}$.

Hence, $\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$ ■

Scratch Work: We want $\frac{1}{(x-1)^2} > \mathbf{M}$ whenever $0 < |x-1| < \delta$

Observe: $\frac{1}{(x-1)^2} = \frac{1}{\left(\sqrt{(x-1)^2}\right)^2} = \frac{1}{(|x-1|)^2} > \frac{1}{\delta^2} = \mathbf{M}$

i.e., $\frac{1}{\delta^2} = \mathbf{M} \quad \Rightarrow \quad \frac{1}{\mathbf{M}} = \delta^2 \quad \Rightarrow \quad \frac{1}{\sqrt{\mathbf{M}}} = \delta$

So let $\delta = \frac{1}{\sqrt{\mathbf{M}}}$

17. **Prove:** $\lim_{x\to 2}(x^2+3x-4)=6$

Proof. Let $\varepsilon > 0$ be given.

Let $\delta = \delta(\varepsilon) > 0$ be given by $\delta = \underline{\min\left(\frac{\varepsilon}{8}, 1\right)}$

Observe: For $\underbrace{0 < |x-2| < \delta}_{0<|x-c|<\delta}$ we have:

$$\underbrace{\left|\left(x^2+3x-4\right)-6\right|}_{|f(x)-L|} = |x^2+3x-10| = |(x+5)(x-2)| = \underbrace{|x+5|}_{<8}\underbrace{|x-2|}_{<\delta} < 8\delta$$

$$\leq 8\left(\tfrac{\varepsilon}{8}\right) = \varepsilon$$

i.e., $\forall\varepsilon > 0$, $\exists\delta = \delta(\varepsilon) > 0$ such that $0 < |x-2| < \delta \Rightarrow |(x^2+3x-4)-6| < \varepsilon$

Hence, $\lim_{x\to 2}(x^2+3x-4)=6$ ■

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Scratch Work:

Given $\varepsilon > 0$, **we want** $\delta = \delta(\varepsilon) > 0$ **such that** $0 < |x-2| < \delta \Rightarrow |(x^2+3x-4)-6| < \varepsilon.$

Observe: $|(x^2+3x-4)-6| = |x^2+3x-10| = |(x+5)(x-2)| = |x+5|\underbrace{|x-2|}_{<\delta}$

What about $|x+5|$ **???** **Without loss of generality, we can assume that** $\delta \leq 1.$

i.e., $|x-2| < 1$

$\Rightarrow -1 < x-2 < 1 \Rightarrow 6 < x+5 < 8$

$\Rightarrow |x+5| < 8$

Thus, we have:

$$|(x^2+3x-4)-6| = \ldots = |(x+5)(x-2)| = \underbrace{|x+5|}_{<8}\underbrace{|x-2|}_{<\delta} < 8\delta \leq 8\left(\tfrac{\varepsilon}{8}\right) = \varepsilon$$

So we let $\delta = \left(\frac{\varepsilon}{8}\right)$. **(Actually, since we assumed that** $\delta \leq 1$, **we let** $\delta = \min\left(\frac{\varepsilon}{8}, 1\right)$**).**

(End of Scratch Work)

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18. **Prove:** $\lim_{x\to 2}(x^3+4)=12$

Proof. Let $\varepsilon>0$ be given.

Let $\delta=\delta(\varepsilon)>0$ be given by $\underline{\delta=\min\left(\frac{\varepsilon}{19},1\right)}$

Observe: For $\underbrace{0<|x-2|<\delta}_{0<|x-c|<\delta}$ we have:

$$\underbrace{\left|\left(x^3+4\right)-12\right|}_{|f(x)-L|}=|x^3-8|=|(x-2)(x^2+2x+4)|=\underbrace{|x-2|}_{<\delta}\underbrace{\left|x^2+2x+4\right|}_{<19}<19\delta$$

$$\leq 19\left(\frac{\varepsilon}{19}\right)=\varepsilon$$

i.e., $\forall\varepsilon>0,\ \exists\delta=\delta(\varepsilon)>0$ such that $0<|x-2|<\delta\Rightarrow|(x^3+4)-12|<\varepsilon$

Hence, $\lim_{x\to 2}(x^3+4)=12$ ■

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Scratch Work:

Given $\varepsilon>0$, we want $\delta=\delta(\varepsilon)>0$ such that $0<|x-2|<\delta\Rightarrow|(x^3+4)-12|<\varepsilon$.

Observe: $|(x^3+4)-12|=|x^3-8|=|(x-2)(x^2+2x+4)|=\underbrace{|x-2|}_{<\delta}|x^2+2x+4|$

What about $|x^2+2x+4|$??? Without loss of generality, we can assume that $\delta\leq 1$.

i.e., $|x-2|<1$

$\Rightarrow -1<x-2<1\Rightarrow 1<x<3$

Hence: $|x|<3$ and $|x^2|<9$

Hence, $|x^2+2x+4|\leq|x^2|+|2x|+|4|<9+2(3)+4=19$

i.e., $|x^2+2x+4|<19$

Thus, we have:

$$|(x^3-5)-3|=\ldots=|(x-2)(x^2+2x+4)|=\underbrace{|x-2|}_{<\delta}\underbrace{\left|x^2+2x+4\right|}_{<19}<19\delta\leq 19\left(\frac{\varepsilon}{19}\right)=\varepsilon$$

So we let $\delta=\min\left(\frac{\varepsilon}{19},1\right)$.

(End of Scratch Work)

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19. **Prove:** $\lim_{x\to 2}(x^3-5)=3$

Proof. Let $\varepsilon>0$ be given.

Let $\delta=\delta(\varepsilon)>0$ be given by $\delta=\underline{\min\left(\frac{\varepsilon}{19},1\right)}$

Observe: For $\underbrace{0<|x-2|<\delta}_{|x-c|<\delta}$ we have:

$$\underbrace{\left|\left(x^3-5\right)-3\right|}_{|f(x)-L|}=\left|x^3-8\right|=\left|(x-2)\left(x^2+2x+4\right)\right|=\underbrace{|x-2|}_{<\delta}\underbrace{\left|x^2+2x+4\right|}_{<19}<19\delta$$

$$\leq 19\left(\frac{\varepsilon}{19}\right)=\varepsilon$$

i.e., $\forall\varepsilon>0$, $\exists\delta=\delta(\varepsilon)>0$ such that $0<|x-2|<\delta\Rightarrow\left|\left(x^3-5\right)-3\right|<\varepsilon$

Hence, $\lim_{x\to 2}(x^3-5)=3$ ■

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Scratch Work:

Given $\varepsilon>0$, we want $\delta=\delta(\varepsilon)>0$ such that $0<|x-2|<\delta\Rightarrow\left|\left(x^3-5\right)-3\right|<\varepsilon$.

Observe: $\left|\left(x^3-5\right)-3\right|=\left|x^3-8\right|=\left|(x-2)\left(x^2+2x+4\right)\right|=\underbrace{|x-2|}_{<\delta}\left|x^2+2x+4\right|$

What about $\left|x^2+2x+4\right|$??? Without loss of generality, we can assume that $\delta\leq 1$.

i.e., $|x-2|<1$

$\Rightarrow -1<x-2<1\Rightarrow 1<x<3$

Hence: $|x|<3$ and $\left|x^2\right|<9$

Hence, $\left|x^2+2x+4\right|<|9+2(3)+4|=19$

i.e., $\left|x^2+2x+4\right|<19$

Thus, we have:

$$\left|\left(x^3-5\right)-3\right|=\ldots=\left|(x-2)\left(x^2+2x+4\right)\right|=\underbrace{|x-2|}_{<\delta}\underbrace{\left|x^2+2x+4\right|}_{<19}<19\delta\leq 19\left(\frac{\varepsilon}{19}\right)=\varepsilon$$

So we let $\delta=\min\left(\frac{\varepsilon}{19},1\right)$.

(End of Scratch Work)

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20. **Prove:** $\lim_{x\to 1} \frac{x+2}{(x-1)^2} = \infty$

Proof. Let $M > 0$ be given.

Let $\delta = \delta(M) > 0$ be given by $\delta = \underline{\min\left(\sqrt{\frac{2}{M}}, 1\right)}$

Observe: For $\underbrace{0 < |x-1| < \delta}_{|x-c|<\delta}$ we have:

$$\underbrace{\frac{x+2}{(x-1)^2}}_{f(x)} = \frac{x+2}{\left(\sqrt{(x-1)^2}\right)^2} = \frac{x+2}{|x-1|^2} > \frac{x+2}{\delta^2} > \frac{2}{\delta^2} \geq \frac{2}{\left(\sqrt{\frac{2}{M}}\right)^2} = \frac{2}{\left(\frac{2}{M}\right)} = M$$

i.e., $\forall M > 0$, $\exists \delta = \delta(M) > 0$ such that $0 < |x-1| < \delta \Rightarrow \frac{x+2}{(x-1)^2} > M$

Hence, $\lim_{x\to 1} \frac{x+2}{(x-1)^2} = \infty$ ■

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Scratch Work:

Given $M > 0$, we want $\delta = \delta(M) > 0$ such that $\underbrace{0 < |x-1| < \delta}_{|x-c|<\delta} \Rightarrow \underbrace{\frac{x+2}{(x-1)^2} > M}_{f(x)>M}$.

Observe: $\frac{x+2}{(x-1)^2} = \frac{x+2}{\left(\sqrt{(x-1)^2}\right)^2} = \frac{x+2}{|x-1|^2} > \frac{x+2}{\delta^2}$

What do we do about x ??? Without loss of generality, we can assume that $\delta \leq 1$.

i.e., $|x-1| < 1$

$\Rightarrow -1 < x-1 < 1 \Rightarrow 2 < x+2 < 4$

i.e., $x > 0$

$\Rightarrow x+2 > 2$

Thus, we have: $\frac{x+2}{(x-1)^2} = \frac{x+2}{\left(\sqrt{(x-1)^2}\right)^2} = \frac{x+2}{|x-1|^2} > \frac{x+2}{\delta^2} > \underbrace{\frac{2}{\delta^2} = M}_{\text{make } \frac{2}{\delta^2} \text{ equal to } M}$

i.e., $\frac{2}{\delta^2} = M \Rightarrow \frac{2}{M} = \delta^2 \Rightarrow \sqrt{\frac{2}{M}} = \delta$

So we let $\delta = \min\left(\sqrt{\frac{2}{M}}, 1\right)$.

(End of Scratch Work)

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21. **Prove:** $\lim_{x\to 1} \frac{x-2}{(x-1)^2} = -\infty$

Proof. Let $M < 0$ be given.

Let $\delta = \delta(M) > 0$ be given by $\underline{\delta = \min\left(\sqrt{\frac{-2}{M}}, 1\right)}$

Observe: For $\underbrace{0 < |x-1| < \delta}_{0<|x-c|<\delta}$ we have:

$$\underbrace{\frac{x-2}{(x-1)^2}}_{f(x)} = \frac{x-2}{\left(\sqrt{(x-1)^2}\right)^2} = \frac{x-2}{|x-1|^2} < \frac{x-2}{\delta^2} < \frac{-2}{\delta^2} \leq \frac{-2}{\left(\sqrt{\frac{-2}{M}}\right)^2} = \frac{-2}{\left(\frac{-2}{M}\right)} = M$$

i.e., $\forall M < 0$, $\exists \delta = \delta(M) > 0$ such that $|x-1| < \delta \Rightarrow \frac{x-2}{(x-1)^2} < M$

Hence, $\lim_{x\to 1} \frac{x-2}{(x-1)^2} = -\infty$ ■

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Scratch Work:

Given $M < 0$, we want $\delta = \delta(M) > 0$ such that $\underbrace{0 < |x-1| < \delta}_{|x-c|<\delta} \Rightarrow \underbrace{\frac{x-2}{(x-1)^2} < M}_{f(x)<M}$.

Observe: $\frac{x-2}{(x-1)^2} = \frac{x-2}{\left(\sqrt{(x-1)^2}\right)^2} = \frac{x-2}{|x-1|^2} < \frac{x-2}{\delta^2}$

What do we do about x ??? Without loss of generality, we can assume that $\delta < 1$.

i.e., $|x-1| < 1$

$\Rightarrow -1 < x-1 < 1 \Rightarrow -2 < x-2 < 0$

Observe: ($x-2$ is *greater* than -2, but $x-2$ is *negative* as well)

Thus, we have: $\frac{x-2}{(x-1)^2} = \frac{x-2}{\left(\sqrt{(x-1)^2}\right)^2} = \frac{x-2}{|x-1|^2} < \frac{x-2}{\delta^2} < \underbrace{\frac{-2}{\delta^2} = M}_{\text{make } \frac{-2}{\delta^2} \text{equal to } M}$

i.e., $\frac{-2}{\delta^2} = M \Rightarrow \frac{-2}{M} = \delta^2 \Rightarrow \sqrt{\frac{-2}{M}} = \delta$

So we let $\delta = \min\left(\sqrt{\frac{-2}{M}}, 1\right)$.

(End of Scratch Work)
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22. **Prove:** $\lim_{x\to 3^+} \frac{x+2}{x-3} = \infty$

Hint: $x \to 3^+$ means that x approaches 3 from the right only. Thus, $(x-3) > 0$.

So instead of $|x-3| < \delta$, we require that $0 < x-3 < \delta$.

Proof. Let $M > 0$ be given.

Let $\delta = \delta(M) > 0$ be given by $\delta = \underline{\frac{2}{M}}$

Observe: For $\underbrace{0 < (x-3) < \delta}_{0<(x-c)<\delta}$ we have:

$$\underbrace{\frac{x+2}{x-3}}_{f(x)} > \frac{x+2}{\delta} > \frac{2}{\delta} \geq \frac{2}{\left(\frac{2}{M}\right)} = M$$

i.e., $\forall M > 0$, $\exists \delta = \delta(M) > 0$ such that $0 < (x-1) < \delta \Rightarrow \frac{x+2}{x-3} > M$

Hence, $\lim_{x\to 3^+} \frac{x+2}{x-3} = \infty$ ■

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Scratch Work:

Given $M > 0$, we want $\delta = \delta(M) > 0$ such that $\underbrace{0 < (x-3) < \delta}_{0<(x-c)<\delta} \Rightarrow \underbrace{\frac{x+2}{x-3} > M}_{f(x)>M}$.

Observe: $\frac{x+2}{x-3} > \frac{x+2}{\delta}$

What do we do about $x+2$???

Recall: $0 < (x-3) < \delta$

$\Rightarrow 3 < x < \delta + 3 \Rightarrow 2 < x < 4$

In particular, $x > 0$

Consequently, $x + 2 > 2$

Thus, we have: $\frac{x+2}{x-3} > \frac{x+2}{\delta} > \underbrace{\frac{2}{\delta} = M}_{\text{make } \frac{2}{\delta} \text{equal to } M}$

i.e., $\frac{2}{\delta} = M \Rightarrow \frac{2}{M} = \delta$

So we let $\delta = \frac{2}{M}$.

(End of Scratch Work)

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23. **Prove:** $\lim_{x\to 3^-} \frac{x+2}{x-3} = -\infty$

Hint: $x \to 3^-$ means that x approaches 3 from the left only. Thus, $(x-3) < 0$.

So instead of $|x-3| < \delta$, we require that $-\delta < x - 3 < 0$.

Proof. Let $M < 0$ be given.

Let $\delta = \delta(M) > 0$ be given by $\delta = \underline{\min\left(-\frac{2}{M}, 1\right)}$

Observe: For $\underbrace{-\delta < (x-3) < 0}_{-\delta<(x-c)<0}$ we have:

$$\underbrace{\frac{x+2}{x-3}}_{f(x)} < \frac{x+2}{-\delta} < \frac{2}{-\delta} \leq \frac{2}{-\left(-\frac{2}{M}\right)} = M$$

i.e., $\forall M < 0$, $\exists \delta = \delta(M) > 0$ such that $-\delta < (x-3) < 0 \Rightarrow \frac{x+2}{x-3} < M$

Hence, $\lim_{x\to 3^-} \frac{x+2}{x-3} = -\infty$ ■

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Scratch Work:

Given $M < 0$, we want $\delta = \delta(M) > 0$ such that $\underbrace{-\delta < (x-3) < 0}_{-\delta<(x-c)<0} \Rightarrow \underbrace{\frac{x+2}{x-3} < M}_{f(x)<M}$.

Observe: $\frac{x+2}{x-3} < \frac{x+2}{-\delta}$

What do we do about x ??? Without loss of generality, we can assume that $\delta < 1$.

i.e., $-1 < (x-3) < 0$

$\Rightarrow 2 < x < 3$

In particular, $x > 0$

$\Rightarrow x + 2 > 2$ ($x+2$ is *greater* than 2, and $x+2$ is *positive* as well)

Thus, we have: $\frac{x+2}{x-3} < \frac{x+2}{-\delta} < \underbrace{\frac{2}{-\delta}}_{\text{make } \frac{2}{-\delta} \text{equal to } M} = M$

i.e., $\frac{2}{-\delta} = M \Rightarrow \frac{2}{M} = -\delta \Rightarrow \delta = -\frac{2}{M}$

So we let $\delta = -\frac{2}{M}$.

(End of Scratch Work)

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24. **Prove:** $\lim_{x\to\infty} \frac{3x+2}{x+5} = 3$

Proof. Let $\varepsilon > 0$ be given.

Let $M = M(\varepsilon) > 0$ be given by $M = \underline{\frac{13}{\varepsilon}}$

Observe: For $\underbrace{x > M}_{x>M}$ we have:

$$\underbrace{\left|\frac{3x+2}{x+5} - 3\right|}_{|f(x)-L|} = \left|\frac{3x+2}{x+5} - \frac{3(x+5)}{x+5}\right| = \left|\frac{3x+2}{x+5} - \frac{3x+15}{x+5}\right| = \left|\frac{-13}{x+5}\right| = \frac{13}{x+5} < \frac{13}{x} < \frac{13}{M} = \frac{13}{\left(\frac{13}{\varepsilon}\right)} = \varepsilon$$

i.e., $\forall \varepsilon > 0$, $\exists M = M(\varepsilon) > 0$ such that $x > M \Rightarrow \left|\frac{3x+2}{x+5} - 3\right| < \varepsilon$

Hence, $\lim_{x\to\infty} \frac{3x+2}{x+5} = 3$ ■

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Scratch Work:

Given $\varepsilon > 0$, we want $M = M(\varepsilon) > 0$ such that $x > M \Rightarrow \left|\frac{3x+2}{x+5} - 3\right| < \varepsilon$.

Observe: $\left|\frac{3x+2}{x+5} - 3\right| = \left|\frac{3x+2}{x+5} - \frac{3(x+5)}{x+5}\right| = \left|\frac{3x+2}{x+5} - \frac{3x+15}{x+5}\right| = \left|\frac{-13}{x+5}\right| = \frac{13}{x+5}$

What about $x + 5$??? Without loss of generality, we can assume that $x > 0$.

Also note that: $x < x + 5$

Thus, we have:

$$\left|\frac{3x+2}{x+5} - 3\right| = \ldots = \frac{13}{x+5} < \frac{13}{x} < \underbrace{\frac{13}{M}}_{\text{make } \frac{13}{M} \text{equal to } \varepsilon} = \varepsilon$$

i.e., $\frac{13}{M} = \varepsilon \Rightarrow M = \frac{13}{\varepsilon}$

So we let $M = \frac{13}{\varepsilon}$.

(End of Scratch Work)

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25. Problems similar to 13 - 24.