# Number Theory Homework Set 1.2

Spring 2015

Pat Rossi Name ______________________

1. ~

   (a) $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$

   **Proof.**

   Observe:

   $$\binom{n}{k}\binom{k}{r} = \frac{n!}{k!(n-k)!}\frac{k!}{r!(k-r)!} = \frac{n!}{(n-k)!}\frac{1}{r!(k-r)!} = \frac{n!}{r!}\frac{1}{(k-r)!(n-k)!} = \frac{n!}{r!}\frac{(n-r)!}{(n-r)!}\frac{1}{(k-r)!(n-k)!}$$

   $$= \frac{n!}{r!(n-r)!}\frac{(n-r)!}{(k-r)!(n-k)!} = \frac{n!}{r!(n-r)!}\frac{(n-r)!}{(k-r)![(n-r)-(k-r)]!} = \binom{n}{r}\binom{n-r}{k-r}$$

   i.e. $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ ■

   (b) $\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$

   **Proof.**

   From part 1.a we have: $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$.

   Letting $r = k-1$, this becomes:

   $$\binom{n}{k}\binom{k}{k-1} = \binom{n}{k-1}\binom{n-(k-1)}{k-(k-1)} \Rightarrow \binom{n}{k}k = \binom{n}{k-1}\binom{n-k+1}{1} \Rightarrow \binom{n}{k}k = \binom{n}{k-1}(n-k+1)$$

   $$\Rightarrow \binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$ ■

2. ~If $2 \leq k \leq n-2$, show that

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$$

**Proof.**

Observe: $\binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k} = \binom{n-2}{k-2} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k}$

Applying Pascal's Rule multiple times, we have:

$$= \underbrace{\binom{n-2}{k-2} + \binom{n-2}{k-1}}_{=\binom{n-1}{k-1}} + \underbrace{\binom{n-2}{k-1} + \binom{n-2}{k}}_{=\binom{n-1}{k}} = \underbrace{\binom{n-1}{k-1} + \binom{n-1}{k}}_{=\binom{n}{k}} = \binom{n}{k}$$

i.e., $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$ ■

3. For $n \geq 1$, derive each of the identities below:

(a) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n} = 2^n$ (Here we use the Binomial Theorem, with $a = b = 1$)

**Proof.**

Observe: $2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = \sum_{i=0}^{n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$

i.e., $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$ ■

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0.$

**Proof.**

Again, we use the Binomial Theorem, letting $a = 1$, and $b = -1$.

Observe: $0 = (1-1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} (-1)^i = \sum_{i=0}^{n} \binom{n}{i} (-1)^i$

$$= (-1)^0 \binom{n}{0} + (-1)^1 \binom{n}{1} + (-1)^2 \binom{n}{2} + \ldots + (-1)^n \binom{n}{n}$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n}$$

Hence, $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0.$ ■

(c) $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} = n2^{n-1}$ [Hint: $n\binom{n-1}{k} = (k+1)\binom{n}{k+1}$]

**Proof.** From the Binimial Theorem with $a = b = 1$, we have:

$$2^{n-1} = (1+1)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} 1^{(n-1)-i} 1^i$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} = \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \ldots + \binom{n-1}{n-1}$$

i.e., $2^{n-1} = \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \ldots + \binom{n-1}{n-1}$

Multiplying both sides by $n$, we have:

$$n2^{n-1} = n\binom{n-1}{0} + n\binom{n-1}{1} + n\binom{n-1}{2} + \ldots + n\binom{n-1}{n-1}$$

$$= 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n}$$

i.e., $n2^{n-1} = 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n}$

Hence, $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} = n2^{n-1}$ ■

**Alternate Proof** for 3(a)

From Part 3.a, $\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \ldots + \binom{n-1}{n-1} = 2^{n-1}$.

Multiplying both sides by $n$, we have:

$n\left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \ldots + \binom{n-1}{n-1}\right] = n2^{n-1}$

$\Rightarrow n\binom{n-1}{0} + n\binom{n-1}{1} + n\binom{n-1}{2} + \ldots + n\binom{n-1}{n-1} = n2^{n-1}$

$\Rightarrow$ (by our hint)

$(0+1)\binom{n}{0+1} + (1+1)\binom{n}{1+1} + (2+1)\binom{n}{2+1} + \ldots + [(n-1)+1]\binom{n}{(n-1)+1} = n2^{n-1}$

$\Rightarrow \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} = n2^{n-1}$ ■

(d) $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^n\binom{n}{n} = 3^n$

**Proof.**

This is just a direct application of the Binomial Theorem with $a = 1$ and $b = 2$.

Observe:

$3^n = (1+2)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 2^i = \sum_{i=0}^{n} \binom{n}{i} 2^i = 2^0\binom{n}{0} + 2^1\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^n\binom{n}{n}$

i.e., $3^n = \binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^n\binom{n}{n}$ ■

(e) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = 2^{n-1}$.

**Proof.**

From part 3.a we have: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$

From part 3.b we have: $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$.

Adding the two, we have:

$$\begin{array}{ccccccccccccc} & \binom{n}{0} & + & \binom{n}{1} & + & \binom{n}{2} & + & \binom{n}{3} & + & \ldots & + & \binom{n}{n} & = & 2^n \\ + & \binom{n}{0} & - & \binom{n}{1} & + & \binom{n}{2} & - & \binom{n}{3} & + & \ldots & + & (-1)^n \binom{n}{n} & = & 0 \\ \hline & 2\binom{n}{0} & & & + & 2\binom{n}{2} & & & + & \ldots & & & = & 2^n \end{array}$$

$\Rightarrow \binom{n}{0} + \binom{n}{2} + \ldots = 2^{n-1}$

Subtracting, part 3.b from part 3.a, we have:

$$\begin{array}{ccccccccccccc} & \binom{n}{0} & + & \binom{n}{1} & + & \binom{n}{2} & + & \binom{n}{3} & + & \ldots & + & \binom{n}{n} & = & 2^n \\ - & \binom{n}{0} & - & \binom{n}{1} & + & \binom{n}{2} & - & \binom{n}{3} & + & \ldots & + & (-1)^n \binom{n}{n} & = & 0 \\ \hline & & & 2\binom{n}{1} & & & + & 2\binom{n}{3} & + & \ldots & & & = & 2^n \end{array}$$

$\Rightarrow \binom{n}{1} + \binom{n}{3} + \ldots = 2^{n-1}$

Hence, $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = 2^{n-1}$. ■

(f) $\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \ldots + \frac{(-1)^n}{n+1}\binom{n}{n} = \frac{1}{n+1}$

**Proof.**

Hint: The left hand side equals: $\frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \ldots + (-1)^n \binom{n+1}{n+1}\right]$

Note that by **Part b** we have: $\binom{n+1}{0} - \binom{n+1}{1} + \binom{n+1}{2} - \binom{n+1}{3} + \ldots + (-1)^{n+1}\binom{n+1}{n+1} = 0$

$\Rightarrow -\binom{n+1}{0} + \binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \ldots + (-1)^n \binom{n+1}{n+1} = 0$

$\Rightarrow \binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \ldots + (-1)^n \binom{n+1}{n+1} = \binom{n+1}{0}$

$\Rightarrow \binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \ldots + (-1)^n \binom{n+1}{n+1} = 1$

Thus, $\frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \ldots + (-1)^n \binom{n+1}{n+1}\right] = \frac{1}{n+1}$

Now we can apply the hint to get our result:

$\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \ldots + \frac{(-1)^n}{n+1}\binom{n}{n} = \frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \ldots + (-1)^n \binom{n+1}{n+1}\right]$

$= \frac{1}{n+1}$

i.e., $\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \ldots + \frac{(-1)^n}{n+1}\binom{n}{n} = \frac{1}{n+1}$ ■

4. Prove the following, for $n \geq 1$ :

(a) $\binom{n}{r} < \binom{n}{r+1}$ if and only if $0 \leq r < \frac{1}{2}(n-1)$

**Proof.**

Observe: $\binom{n}{r} < \binom{n}{r+1} \Leftrightarrow \frac{n!}{r!(n-r)!} < \frac{n!}{(r+1)!(n-r-1)!} \Leftrightarrow \frac{1}{r!(n-r)!} < \frac{1}{(r+1)!(n-r-1)!}$

$\Leftrightarrow \frac{(r+1)!}{r!} < \frac{(n-r)!}{(n-r-1)!} \Leftrightarrow r+1 < n-r \Leftrightarrow 2r < n-1 \Leftrightarrow r < \frac{1}{2}(n-1)$

$\Leftrightarrow r < \frac{1}{2}(n-1) \Leftrightarrow 0 < r < \frac{1}{2}(n-1)$ (because $r \geq 0$ in order for $\binom{n}{r}$ to be defined.)

Hence, $\binom{n}{r} < \binom{n}{r+1}$ if and only if $0 \leq r < \frac{1}{2}(n-1)$. ■

(b) $\binom{n}{r} > \binom{n}{r+1}$ if and only if $(n-1) \geq r > \frac{1}{2}(n-1)$.

**Proof.**

Observe: $\binom{n}{r} > \binom{n}{r+1} \Leftrightarrow \frac{n!}{r!(n-r)!} > \frac{n!}{(r+1)!(n-r-1)!} \Leftrightarrow \frac{1}{r!(n-r)!} > \frac{1}{(r+1)!(n-r-1)!}$

$\Leftrightarrow \frac{(r+1)!}{r!} > \frac{(n-r)!}{(n-r-1)!} \Leftrightarrow r+1 > n-r \Leftrightarrow 2r > n-1 \Leftrightarrow r > \frac{1}{2}(n-1)$

$\Leftrightarrow (n-1) \geq r > \frac{1}{2}(n-1)$ (because $n \geq (r+1)$ in order for $\binom{n}{r+1}$ to be defined, which in turn implies that $(n-1) \geq r$.)

Hence, $\binom{n}{r} > \binom{n}{r+1}$ if and only if $(n-1) \geq r > \frac{1}{2}(n-1)$. ■

(c) $\binom{n}{r} = \binom{n}{r+1}$ if and only if $n$ is an odd integer, and $r = \frac{1}{2}(n-1)$

**Proof.**

Observe:

$\binom{n}{r} = \binom{n}{r+1} \Leftrightarrow \frac{n!}{r!(n-r)!} = \frac{n!}{(r+1)!(n-r-1)!} \Leftrightarrow \frac{1}{r!(n-r)!} = \frac{1}{(r+1)!(n-r-1)!}$

$\Leftrightarrow \frac{(r+1)!}{r!} = \frac{(n-r)!}{(n-r-1)!} \Leftrightarrow r+1 = n-r \Leftrightarrow 2r = n-1 \Leftrightarrow r = \frac{1}{2}(n-1)$

Note, that since $2r = n-1$, it follows that $n-1$ is even. Therefore $n$ is odd. ■

5. ~

(a) For $n \geq 2$, prove that $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}$ (Hint: Use induction and Pascal's Rule.)

**Proof.**

**Step 1** Show that the proposition holds true for $n = 2$.

The proposition holds for $n = 2$, as $\binom{2}{2} = \binom{2+1}{3}$

**Step 2** Assume that the proposition holds for $n = k$, and show that this implies that the proposition also holds for $n = k + 1$.

i.e., assume that $\underbrace{\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{k}{2} = \binom{k+1}{3}}_{\text{induction hypothesis}}$

and show that $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{k+1}{2} = \binom{(k+1)+1}{3}$

**Observe:**

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{k}{2} + \binom{k+1}{2} = \underbrace{\binom{k+1}{3}}_{\substack{\text{By induction}\\\text{hypothesis}}} + \binom{k+1}{2}$$

$$= \underbrace{\binom{k+2}{3}}_{\substack{\text{By Pascal's}\\\text{Rule}}} = \binom{(k+1)+1}{3}$$

i.e., $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{k+1}{2} = \binom{(k+1)+1}{3}$

Hence, $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}$ for all natural numbers, $n \geq 2$. ∎

(b) From Part (a) and the relations $m^2 = 2\binom{m}{2} + m$, deduce the formula

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

**Proof.** Using the identity: $m^2 = 2\binom{m}{2} + m$, we have:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = 1^2 + \left(2\binom{2}{2} + 2\right) + \left(2\binom{3}{2} + 3\right) + \ldots + \left(2\binom{n}{2} + n\right)$$

$$= 2\left[\binom{2}{2} + \binom{3}{2} + \ldots + \binom{n}{2}\right] + (1 + 2 + 3 + \ldots + n)$$

$$= 2 \cdot \underbrace{\binom{n+1}{3}}_{\text{from Part a}} + (1 + 2 + 3 + \ldots + n) = 2 \cdot \binom{n+1}{3} + \underbrace{\frac{n(n+1)}{2}}_{\substack{\text{From Sec 1.1} \\ \text{Exercise 1(a)}}} =$$

$$\frac{2(n+1)!}{3![(n+1)-3]!} + \frac{n(n+1)}{2} = \frac{2(n+1)!}{3!(n-2)!} + \frac{n(n+1)}{2} = \frac{2(n+1)n(n-1)(n-2)!}{6(n-2)!} + \frac{n(n+1)}{2}$$

$$= \frac{2(n+1)n(n-1)}{6} + \frac{3}{3}\frac{n(n+1)}{2} = \frac{(n+1)n[2(n-1)+3]}{6} = \frac{(n+1)n(2n+1)}{6}$$

i.e., $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ ■

(c) Apply the formula in Part (a) to obtain a proof that

$$1 \cdot 2 + 2 \cdot 3 + \ldots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

Hint: observe that $(m-1)m = 2\binom{m}{2}$.

**Proof.** Using the identity: $(m-1)m = 2\binom{m}{2}$, we have:

$$\underbrace{1 \cdot 2}_{m=2} + \underbrace{2 \cdot 3}_{m=3} + \ldots + \underbrace{n \cdot (n+1)}_{m=n+1} = 2\binom{2}{2} + 2\binom{3}{2} + \ldots + 2\binom{n}{2} + 2\binom{n+1}{2}$$

$$= \underbrace{2\binom{(n+1)+1}{3}}_{\text{By Part a}} = 2\binom{n+2}{3}$$

i.e.,$1 \cdot 2 + 2 \cdot 3 + \ldots + n \cdot (n+1) = 2\binom{n+2}{3}$

$$\Rightarrow 1 \cdot 2 + 2 \cdot 3 + \ldots + n \cdot (n+1) = 2\binom{n+2}{3} = \frac{2(n+2)!}{3![(n+2)-3]!} = \frac{2(n+2)!}{3!(n-1)!} = \frac{2(n+2)(n+1)n(n-1)!}{3!(n-1)!}$$

$$= \frac{2(n+2)(n+1)n}{3 \cdot 2 \cdot 1} = \frac{n(n+1)(n+2)}{3}$$

i.e., $1 \cdot 2 + 2 \cdot 3 + \ldots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$ ■