MTH 4436 HW Set 1.1

Fall 2016

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1. ~

   (a) $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$

   i.e., $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

   Step 1

   First, show that the proposition is true for $n = 1$

   $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$ (this is true.)

   Step 2

   Next, assume that the proposition is true for $n = k$, and show that this implies that the proposition is true for $n = k + 1$

   i.e., assume that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ and show that $\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$

   i.e., show that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

   Observe: $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

   $$= \frac{(k+1)(k+2)}{2}$$

   i.e., $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

   Hence, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \geq 1$. ■

(b) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for all $n \geq 1$

i.e., $\sum_{i=1}^{n} (2i - 1) = n^2$

Step 1

First, show that the proposition is true for $n = 1$

$\sum_{i=1}^{1} (2i - 1) = (2(1) - 1) = 1 = 1^2$ (this is true.)

Step 2

Next, assume that the proposition is true for $n = k$, and show that this implies that the proposition is true for $n = k + 1$

i.e., assume that $\sum_{i=1}^{k} (2i - 1) = k^2$ and show that $\sum_{i=1}^{k+1} (2i - 1) = (k + 1)^2$

Observe: $\sum_{i=1}^{k+1} (2i - 1) = \left(\sum_{i=1}^{k} (2i - 1)\right) + 2(k + 1) - 1 = k^2 + 2(k + 1) - 1$

$$= k^2 + 2k + 1 = (k + 1)^2$$

i.e., $\sum_{i=1}^{k+1} (2i - 1) = (k + 1)^2$.

Hence, $\sum_{i=1}^{n} (2i - 1) = n^2$ for all $n \geq 1$. ■

(c) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n\,(n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \geq 1$

i.e., $\sum_{i=1}^{n} i\,(i+1) = \frac{n(n+1)(n+2)}{3}$

Step 1

First, show that the proposition is true for $n = 1$

$\sum_{i=1}^{1} i\,(i+1) = 1\,(1+1) = 2 = \frac{1(1+1)(1+2)}{3}$ (this is true.)

Step 2

Next, assume that the proposition is true for $n = k$, and show that this implies that the proposition is true for $n = k+1$

i.e., assume that $\sum_{i=1}^{k} i\,(i+1) = \frac{k(k+1)(k+2)}{3}$

and show that $\sum_{i=1}^{k+1} i\,(i+1) = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$

i.e., Show that $\sum_{i=1}^{k+1} i\,(i+1) = \frac{(k+1)(k+2)(k+3)}{3}$

Observe: $\sum_{i=1}^{k+1} i\,(i+1) = \sum_{i=1}^{k} i\,(i+1) + (k+1)\,((k+1)+1)$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)\,((k+1)+1)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)\,(k+2) = (k+1)\,(k+2)\left(\frac{k}{3}+1\right)$$

$$= (k+1)\,(k+2)\left(\frac{k}{3}+\frac{3}{3}\right) = (k+1)\,(k+2)\left(\frac{k+3}{3}\right)$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

i.e., $\sum_{i=1}^{k+1} i\,(i+1) = \frac{(k+1)(k+2)(k+3)}{3}$

Hence, $\sum_{i=1}^{n} i\,(i+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \geq 1$. $\blacksquare$

(d) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all $n \geq 1$

i.e., $\sum_{i=1}^{n} (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Step 1

First, show that the proposition is true for $n = 1$

$\sum_{i=1}^{1} (2i-1)^2 = (2(1)-1)^2 = 1 = \frac{n(2n-1)(2n+1)}{3}$ (this is true.)

Step 2

Next, assume that the proposition is true for $n = k$, and show that this implies that the proposition is true for $n = k+1$

i.e., assume that $\sum_{i=1}^{k} (2i-1)^2 = \frac{k(2k-1)(2k+1)}{3}$

and show that $\sum_{i=1}^{k+1} (2i-1)^2 = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$

i.e., Show that $\sum_{i=1}^{k+1} (2i-1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$

Observe:

$$\sum_{i=1}^{k+1} (2i-1)^2 = \sum_{i=1}^{k} (2i-1)^2 + (2(k+1)-1)^2$$

$$= \frac{k(2k-1)(2k+1)}{3} + (2(k+1)-1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= (2k+1)\left[\frac{k(2k-1)}{3} + (2k+1)\right] = (2k+1)\left[\frac{k(2k-1)}{3} + \frac{3(2k+1)}{3}\right]$$

$$= (2k+1)\left[\frac{2k^2-k+6k+3}{3}\right] = (2k+1)\left[\frac{2k^2+5k+3}{3}\right] = (2k+1)\left[\frac{2k^2+5k+3}{3}\right]$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

i.e., $\sum_{i=1}^{k+1} (2i-1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$

Hence, $\sum_{i=1}^{n} (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all $n \geq 1$ ■

(e) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ for all $n \geq 1$

i.e., $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Step 1

First, show that the proposition is true for $n = 1$

$\sum_{i=1}^{1} i^3 = 1^3 = 1 = \left[\frac{1(1+1)}{2}\right]^2$ this is true.

Step 2

Next, assume that the proposition is true for $n = k$, and show that this implies that the proposition is true for $n = k + 1$

i.e., assume that $\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$

and show that $\sum_{i=1}^{k+1} i^3 = \left[\frac{(k+1)((k+1)+1)}{2}\right]^2$

i.e., Show that $\sum_{i=1}^{k+1} i^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$

Observe:

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{2^2} + (k+1)^3 = (k+1)^2\left[\frac{k^2}{2^2} + (k+1)\right]$$

$$= (k+1)^2\left[\frac{k^2}{2^2} + \frac{4k+4}{2^2}\right] = (k+1)^2\left[\frac{k^2+4k+4}{2^2}\right] = (k+1)^2\left[\frac{(k+2)^2}{2^2}\right] =$$

$$= \frac{(k+1)^2(k+2)^2}{2^2} = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

i.e., $\sum_{i=1}^{k+1} i^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$

Hence, $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$ for all $n \geq 1$ ■

2. $a + ar + ar^2 + \ldots + ar^n = \frac{a\left(r^{n+1}-1\right)}{r-1}$ for all $n \geq 1$

i.e., $\sum_{i=0}^{n} ar^i = \frac{a\left(r^{n+1}-1\right)}{r-1}$ for all $n \geq 1$

Step 1

First, show that the proposition is true for $n = 0$

$\sum_{i=0}^{0} ar^i = ar^0 = a = \frac{a\left(r^{0+1}-1\right)}{r-1}$ (this is true.)

Step 2

Next, assume that $P(k)$ is True, and show that $P(k+1)$ is True.

i.e., assume that $\sum_{i=0}^{k} ar^i = \frac{a\left(r^{k+1}-1\right)}{r-1}$

and show that $\sum_{i=0}^{k+1} ar^i = \frac{a\left(r^{(k+1)+1}-1\right)}{r-1}$

Observe:

$$\sum_{i=0}^{k+1} ar^i = \left(\sum_{i=0}^{k} ar^i\right) + ar^{k+1} = \frac{a\left(r^{k+1}-1\right)}{r-1} + ar^{k+1} = \frac{a\left(r^{k+1}-1\right)}{r-1} + \frac{(r-1)ar^{k+1}}{r-1}$$

$$= \frac{ar^{k+1}-a}{r-1} + \frac{ar^{(k+1)+1}-ar^{k+1}}{r-1} = \frac{ar^{(k+1)+1}-a}{r-1} = \frac{a\left(r^{(k+1)+1}-1\right)}{r-1}$$

i.e., $\sum_{i=0}^{k+1} ar^i = \frac{a\left(r^{(k+1)+1}-1\right)}{r-1}$

Hence, $\sum_{i=0}^{n} ar^i = \frac{a\left(r^{n+1}-1\right)}{r-1}$ for all $n \geq 1$ ■

(See the next page for an alternate proof.)

Alternate Proof:

$$\begin{aligned} \text{Let} \quad S &= a + ar + ar^2 + \ldots + ar^n \\ \text{Then} \quad rS &= ar + ar^2 + ar^3 + \ldots + ar^n + ar^{n+1} \end{aligned}$$

Subtracting the second equation form the first, we have:

$$\begin{array}{rcl} S & = & a + ar + ar^2 + ar^3 + \ldots + ar^n \\ -\quad rS & = & \quad ar + ar^2 + ar^3 + \ldots + ar^n + ar^{n+1} \\ \hline S - rS & = & a \qquad\qquad\qquad\qquad\qquad\qquad - ar^{n+1} \end{array}$$

i.e., $S - rS = a - ar^{n+1}$

Solving for S, we have:

$S - rS = a - ar^{n+1} \Rightarrow S\,(1-r) = a\,(1 - r^{n+1}) \Rightarrow S = \frac{a\left(1-r^{n+1}\right)}{(1-r)} = \frac{a\left(r^{n+1}-1\right)}{(r-1)}$

i.e., $S = \frac{a\left(r^{n+1}-1\right)}{(r-1)}$

Hence, $a + ar + ar^2 + ar^3 + \ldots + ar^n = \frac{a\left(r^{n+1}-1\right)}{(r-1)}$ ■

6. Prove that $n! > n^2$ for every integer $n \geq 4$, whereas $n! > n^3$ for every integer $n \geq 6$.

$P(n) : n! > n^2$ for $n \geq 4$

pf/

Step 1

Show that $P(4)$ is True

Observe: $4! = 24 > 16 = 4^2$

i.e., $4! > 4^2$ (P(4) is true.)

Step 2

Assume that $P(k)$ is True, and show that $P(k+1)$ is True.

i.e., Assume that $k! > k^2$, and show that $(k+1)! > (k+1)^2$

First, note that for $n \geq 4$ (and hence $k \geq 4$), $k^2 > (k+1)$.

This follows from observing that $k^2 = \underbrace{k \cdot k > 2 \cdot k}_{\text{because } k \geq 4 > 2} = (k+k) > (k+1)$

Observe: $(k+1)! = (k+1) \cdot k! > (k+1) \cdot k^2 > (k+1)(k+1) = (k+1)^2$

i.e., $(k+1)! > (k+1)^2$

Hence, $n! > n^2$ for every integer $n \geq 4$. ■

$$\boxed{P(n) : n! > n^3 \text{ for } n \geq 6}$$

pf/

Step 1

Show that $P(6)$ is True

Observe: $6! = 720 > 216 = 6^3$

i.e., $6! > 6^3$ (P(6) is true.)

Step 2

Assume that $P(k)$ is True, and show that $P(k+1)$ is True.

i.e., Assume that $k! > k^3$, and show that $(k+1)! > (k+1)^3$

First, note that for $n \geq 6$ (and hence $k \geq 6$), $k^2 > (2k+1)$.

This follows from observing that $k^2 = \underbrace{k \cdot k > 3 \cdot k}_{\text{because } k \geq 6 > 3} = (2k+k) > (2k+1)$

Observe: $(k+1)! = (k+1) \cdot k! > (k+1) \cdot k^3 = (k+1)(k^2 \cdot k) > (k+1)(k^2 \cdot 2) = (k+1)(k^2 + k^2) > (k+1)(k^2 + 2k + 1) = (k+1)(k+1)^2 = (k+1)^3$

i.e., $(k+1)! > (k+1)^3$

Hence, $n! > n^3$ for every integer $n \geq 6$. $\blacksquare$

7. $1\,(1!) + 2\,(2!) + 3\,(3!) + \ldots + n\,(n!) = (n+1)! - 1$ for all $n \geq 1\,(k+1)$

i.e., $\sum_{i=1}^{n} i\,(i!) = (n+1)! - 1$ for all $n \geq 1$

Step 1

First, show that the proposition is true for $n = 1$

$\sum_{i=1}^{1} i\,(i!) = 1\,(1!) = 1 = (1+1)! - 1$ (this is true.)

Step 2

Next, assume that the proposition is true for $n = k$, and show that this implies that the proposition is true for $n = k+1$

i.e., assume that $\sum_{i=1}^{k} i\,(i!) = (k+1)! - 1$

and show that $\sum_{i=1}^{k+1} i\,(i!) = ((k+1)+1)! - 1$

Observe: $\sum_{i=1}^{k+1} i\,(i!) = \left(\sum_{i=1}^{k} i\,(i!)\right) + (k+1)\,(k+1)! =$

$[(k+1)! - 1] + (k+1)\,(k+1)! = (k+1)! + (k+1)\,(k+1)! - 1 =$

$(k+1)!\,[1 + (k+1)] - 1 = (k+1)!\,[(k+1)+1] - 1 = [(k+1)+1]! - 1$

i.e., $\sum_{i=1}^{k+1} i\,(i!) = [(k+1)+1]! - 1$

Hence, $\sum_{i=1}^{n} i\,(i!) = (n+1)! - 1$ for all $n \geq 1$ ■

9. Establish the Bernoulli Inequality: If $(1+a) > 0$, then $(1+a)^n \geq 1 + na$ for all $n \geq 1$. (This is $P(n)$.)

pf/

Suppose that $(1+a) > 0$

Step 1

First, show that the proposition is true for $n = 1$

Observe: $(1+a)^1 = 1 + a = 1 + (1)a$

i.e., $(1+a)^1 \geq 1 + (1)a$ (this is True.)

Step 2

Next, assume that the proposition is true for $n = k$, and show that this implies that the proposition is true for $n = k + 1$

i.e., assume that $(1+a)^k \geq 1 + ka$ for some $k \in \mathbb{N}$, and show that $(1+a)^{k+1} \geq 1 + (k+1)a$.

Observe: $(1+a)^{k+1} = (1+a)^k(1+a) \geq (1+ka)(1+a) = 1 + ka + a + ka^2$

$$= 1 + (k+1)a + \underbrace{ka^2}_{\geq 0} \geq 1 + (k+1)a$$

i.e., $(1+a)^{k+1} \geq 1 + (k+1)a$

Therefore, $(1+a)^n \geq 1 + na$ for all $n \geq 1$. ■