

Homework - Sec. 2.3, Page 24

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Pat Rossi

Name _____

1. If $a|b$, show that $(-a)|b$, $a|(-b)$, and $(-a)|(-b)$.

Let the hypothesis be given (i.e., suppose that $a|b$).

Then \exists an integer m such that $b = am$.

$$\Rightarrow b = (-a)(-m) \Rightarrow (-a)|b.$$

Also, the fact that $b = am$ implies that $(-b) = a(-m)$, hence, $a|(-b)$.

Finally, the fact that $b = am$ implies that $(-b) = (-a)(m)$, hence, $(-a)|(-b)$. ■

2. Given integers a, b, c, d , verify the following:

- (a) If $a|b$, then $a|bc$.

Let the hypothesis be given (i.e., suppose that $a|b$).

Then \exists an integer m such that $b = am$.

$$\Rightarrow bc = (am)c = a(mc)$$

i.e., $bc = a(mc)$.

Hence, $a|bc$. ■

- (b) If $a|b$ and $a|c$, then $a^2|bc$.

Let the hypothesis be given (i.e., suppose that $a|b$ and $a|c$).

Then \exists an integer(s) m and n such that $b = am$ and $c = an$.

$$\text{Then } bc = (am)(an) = a^2(mn).$$

i.e., $bc = a^2(mn)$, and hence, $a^2|bc$. ■

- (c) $a|b$ if and only if $ac|bc$, where $c \neq 0$.

$$\boxed{a|b \Rightarrow ac|bc}$$

Suppose that $a|b$. Then \exists an integer m such that $b = am$.

$$\text{Observe: } bc = (am)c = (ac)m$$

i.e., $bc = (ac)m$, and hence, $ac|bc$.

$$\boxed{ac|bc \Rightarrow a|b}$$

Suppose that $ac|bc$. Then \exists an integer m such that $bc = (ac)m$.

$$\Rightarrow bc = (am)c.$$

Since $c \neq 0$, we can divide both sides by zero.

$$\Rightarrow b = am.$$

Hence, $a|b$. ■

(d) If $a|b$ and $c|d$, then $ac|bd$

Let the hypothesis be given (i.e., suppose that $a|b$ and $c|d$).

Then \exists an integer(s) m and n such that $b = am$ and $d = cn$.

Then $bd = (am)(cn) = (ac)(mn)$.

i.e., $bd = (ac)(mn)$, and hence, $ac|bd$. ■

3. Prove or disprove: If $a|(b+c)$, then either $a|b$ or $a|c$.

This is false. To show that it's false, we need to offer a counterexample.

Consider: $a = 2$, $b = 3$, and $c = 5$.

$a|(b+c)$, but $a \nmid b$ and $a \nmid c$.

5. Prove that for any integer a , one of the integers, $a, a+2, a+4$ is divisible by 3.

Let a be any integer. Then by the division algorithm, there are exactly three mutually exclusive and mutually exhaustive possibilities — either $a = 3k$, $a = 3k+1$, or $a = 3k+2$.

Case 1: $a = 3k$

If $a = 3k$, then our assertion is proved.

Case 2: $a = 3k+1$

If $a = 3k+1$, then $a+2 = (3k+1)+2 = 3k+3 = 3(k+1)$

i.e., $a+2 = 3(k+1)$. Hence, $3|(a+2)$

Case 2: $a = 3k+2$

If $a = 3k+2$, then $a+4 = (3k+2)+4 = 3k+6 = 3(k+2)$

i.e., $a+4 = 3(k+2)$. Hence, $3|(a+4)$

Since all possibilities have been exhausted and since our proposition is true for each possibility, we can say that for any integer a , one of the integers, $a, a+2, a+4$ is divisible by 3. ■

8. ~

- (a) The sum of the squares of two odd integers cannot be a perfect square.

Two arbitrary odd integers can be represented as $2k + 1$ and $2j + 1$.

The sum of their squares is $(2k + 1)^2 + (2j + 1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 4k^2 + 4k + 4j^2 + 4j + 2 = 4(k^2 + k + j^2 + j) + 2 = 4m + 2$, where $m = k^2 + k + j^2 + j$

The point here, is that the sum of the squares of any two odd numbers is of the form $4m + 2$.

Could this be a perfect square? Well, let's see what form(s) perfect squares can have.

An even number $2n$, when squared, has the form: $4n^2 = 4m$, where $m = n^2$.

An odd number $2n + 1$, when squared, has the form: $4n^2 + 4n + 1 = 4m + 1$, where $m = n^2 + n$.

What we have learned is that perfect squares are either of the form $4m$ or $4m + 1$. Perfect squares are *never* of the form $4m + 2$, which is the form that the sum of the squares of two odd numbers always has.

Hence, the sum of the squares of two odd integers cannot be a perfect square. ■

- (b) The product of four consecutive integers is one less than a perfect square.

We can represent the four consecutive integers as $n, n + 1, n + 2, n + 3$.

Their product is $n(n + 1)(n + 2)(n + 3) = n^4 + 6n^3 + 11n^2 + 6n$

This is one less than $n^4 + 6n^3 + 11n^2 + 6n + 1 = (n^2 + 3n + 1)^2$, which is a perfect square. ■

9. Establish that the difference of two consecutive cubes is never divisible by 2.

Case 1: The smaller number is even.

Let a be even. Then \exists a natural number k such that $a = 2k$. This being the case, $a + 1 = 2k + 1$

The difference of the cubes of these numbers is $(a + 1)^3 - a^3 = (2k + 1)^3 - (2k)^3 = 12k^2 + 6k + 1 = \underbrace{2(6k^2 + 3k)}_{2m+1} + 1$, which is odd.

Case 2: The smaller number is odd.

Let a be odd. Then \exists a natural number k such that $a = 2k + 1$. This being the case, $a + 1 = 2k + 2$

The difference of the cubes of these numbers is $(a + 1)^3 - a^3 = (2k + 2)^3 - (2k + 1)^3 = 12k^2 + 18k + 7 = \underbrace{2(6k^2 + 9k + 3)}_{2m+1} + 1$, which is odd.

Since this exhausts all cases, and each case results in the difference of two consecutive cubes being odd, our assertion is proved. ■

11. If a and b are integers, not both of which are zero, verify that:

$$\gcd(a, b) = \gcd(-a, b) = \gcd(a, -b) = \gcd(-a, -b).$$

$$d = \gcd(a, b)$$

$\Leftrightarrow d$ is the smallest natural number such that $\exists x, y \in \mathbf{Z}$ such that $ax + by = d$

$\Leftrightarrow d$ is the smallest natural number such that $\exists x_1, y \in \mathbf{Z}$ such that

$$\underbrace{(-a)x_1}_{x_1 = -x} + by = d \tag{Eq.1}$$

(and consequently $d = \gcd(-a, b)$)

$\Leftrightarrow d$ is the smallest natural number such that $\exists x_1, y_1 \in \mathbf{Z}$ such that

$$\underbrace{(-a)x_1}_{x_1 = -x} + \underbrace{(-b)y_1}_{y_1 = -y} = d$$

(and consequently $d = \gcd(-a, -b)$)

$\Leftrightarrow d$ is the smallest natural number such that $\exists x, y_1 \in \mathbf{Z}$ such that $\underbrace{(a)x}_{x = -x_1} + \underbrace{(-b)y_1}_{y_1 = -y} = d.$

(and consequently $d = \gcd(a, -b)$ ■)

Remark 1 In Eq. 1, if d were NOT the smallest natural number such that

$$\underbrace{(-a)x_1}_{x_1 = x} + by = d, \text{ then there must be a smaller natural number } c \text{ such that}$$

$$\underbrace{(-a)x_1}_{x_1 = x} + by = c.$$

But this would imply that $\underbrace{ax_1}_{x_1 = x} + by = c$, contradicting the assumption that d was the smallest such natural number.

Remark 2 This was not the solution that I originally had. Julian Allagan showed me this proof and I liked it, so I changed the one that I had.

18a. Prove: The product of any three consecutive integers is divisible by 6.

Let the integers be represented as n , $n + 1$, $n + 2$.

Claim: At least one of the integers must be even.

By the Division Algorithm, n is either of the form $2k$ or $2k + 1$.

If n is of the form $2k$, our claim is proved.

If n is of the form $2k + 1$, then $n + 1 = (2k + 1) + 1 = 2(k + 1)$, and hence, $n + 1$ is even.

End of Claim

Claim: At least one of the integers must be divisible by 3.

By the Division Algorithm, n is either of the form $3k$, $3k + 1$, or $3k + 2$.

If n is of the form $3k$, then our claim is proved.

If n is of the form $3k + 1$, then $n + 2 = (3k + 1) + 2 = 3(k + 1)$, and hence, $n + 2$ is divisible by 3.

If n is of the form $3k + 2$, then $n + 1 = (3k + 2) + 1 = 3(k + 1)$, and hence, $n + 1$ is divisible by 3.

End of Claim

Thus $2 \mid (n)(n + 1)(n + 2)$ and $3 \mid (n)(n + 1)(n + 2)$.

Since $\gcd(2, 3) = 1$, the second corollary to Theorem 2.4 tells us that

$$(2 \cdot 3) \mid (n)(n + 1)(n + 2).$$

i.e., the product of any three consecutive integers is divisible by 6. ■

18b. Prove: The product of any four consecutive integers is divisible by 24.

Let the integers be represented as n , $n + 1$, $n + 2$, and $n + 3$.

By our results in part 11a, the product $(n)(n + 1)(n + 2)(n + 3)$ is divisible by 3.

Claim: Our product is divisible by 8.

By the Division Algorithm, n is either of the form $4k$, $4k + 1$, $4k + 2$, or $4k + 3$.

If n is of the form $4k$, then $4k + 2 = 2(2k + 1)$, and the product $(n)(n + 1)(n + 2)(n + 3) = (4k)(4k + 1)(4k + 2)(4k + 3) = (4k)(4k + 1)2(2k + 1)(4k + 3) = 8k(4k + 1)(2k + 1)(4k + 3)$. Our claim is proved.

If n is of the form $4k + 1$, then $n + 1 = (4k + 1) + 1 = 2(2k + 1)$. Furthermore, $n + 3 = (4k + 1) + 3 = 4(k + 1)$.

Hence, our product $(n)(n + 1)(n + 2)(n + 3) = (4k + 1)(4k + 2)(4k + 3)(4k + 4) = (4k + 1)2(2k + 1)(4k + 3)4(k + 1) = 8(4k + 1)(2k + 1)(4k + 3)(k + 1)$.

And our claim is proved.

Similar arguments can be used to show that if $n = 4k + 2$ or $n = 4k + 3$, then $8 \mid (n)(n + 1)(n + 2)(n + 3)$

End of Claim

We have established that $8 \mid (n)(n + 1)(n + 2)(n + 3)$ and that $3 \mid (n)(n + 1)(n + 2)(n + 3)$

Since $\gcd(3, 8) = 1$, the second corollary to Theorem 2.4 tells us that

$$(3 \cdot 8) \mid (n)(n + 1)(n + 2)(n + 3).$$

i.e., the product of any four consecutive integers is divisible by 24. ■

18c. Prove: The product of any five consecutive integers is divisible by 120.

By our results in part 11b, the product $(n)(n + 1)(n + 2)(n + 3)(n + 4)$ is divisible by 24.

Since $\gcd(5, 24) = 1$, it remains to show that $5 \mid (n)(n + 1)(n + 2)(n + 3)(n + 4)$.

An argument similar to those of part 11a., can be used to show this.

Hence, the product of five consecutive integers is divisible by 120. ■