MTH 4436 Homework Set 2.5 (p. 37)

Spring 2015

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1. Which of the following Diophantine Equations cannot be solved?

   Remark 1 *Recall: The Diophantine equation*

   $$ax + by = c$$

   has a solution exactly when $\gcd(a, b) | c$.

   (a) $6x + 51y = 22$

   $\gcd(6, 51) = 3$

   Since $3 \nmid 22$, the equation has no solution.

   (b) $33x + 14y = 115$

   $\gcd(33, 14) = 1$

   Since $1|115$, the equation has infinitely many solutions.

   (c) $14x + 35y = 93$

   $\gcd(14, 35) = 7$

   Since $7 \nmid 93$, the equation has no solution.

2. Determine ALL solutions in the integers of the following Diophantine equations.

   (a) $56x + 72y = 40$

   First, use the Division Algorithm to find $\gcd(56, 72)$

   $72 = q_1(56) + r_1$
   $72 = (1)(56) + 16$ eq. 2
   Repeat with 56 and 16
   $56 = q_2(16) + r_2$
   $56 = (3)(16) + 8$ eq. 1
   Repeat with 16 and 8
   $16 = q_3(8) + r_3$
   $16 = (2)(8) + 0$
   $\gcd(56, 72)$ is the last non-zero divisor
   $\gcd(56, 72) = 8$

   Since $\gcd(56, 72)\,|40$, the equation $56x + 72y = 40$ has infinitely many solutions.

   First though, we solve the related equation, $56x + 72y = \gcd(56, 72)$.

   (i.e., $56x + 72y = 8$).

   $8 = 56 - (3)(16)$ (From eq. 1)
   $16 = 72 - (1)(56)$ (From eq.2)
   $\Rightarrow 8 = 56 - (3)(72 - (1)(56))$
   $\Rightarrow 8 = (4)(56) - (3)(72)$
   i.e., $56(4) + 72(-3) = 8$

   Thus, $(x_0, y_0) = (4, -3)$ is a particular solution of the equation $56x + 72y = 8$.

   We want the particular solution (x_p, y_p) to the equation $56x + 72y = \underbrace{40}_{c}$

   To get (x_p, y_p), we multiply (x_0, y_0) by $\frac{c}{d} = \frac{40}{8} = 5$

   Thus, $(x_p, y_p) = (20, -15)$ is a particular solution of the Diophantine equation $56x + 72y = 40$

   To get the general solution, we must first find the homogeneous solution (x_h, y_h).

   The homogeneous solution has the form:

   $(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

   i.e., $(x_h, y_h) = \left(\frac{72}{8}t, -\frac{56}{8}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

   i.e., $(x_h, y_h) = (9t, -7t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

Our general solution $(x, y) = (x_p, y_p) + (x_h, y_h)$

$\Rightarrow (x, y) = (20, -15) + (9t, -7t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x, y) = (20 + 9t, -15 - 7t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$\text{i.e., } x = 20 + 9t; \quad y = -15 - 7t \quad \text{for } t \in \mathbf{Z}$$

(b) $24x + 138y = 18$

First, use the Division Algorithm to find $\gcd(24, 138)$

$$
\begin{array}{ll}
138 = q_1 (24) + r_1 & \\
138 = (5)(24) + 18 & \text{eq. 2} \\
\quad \text{Repeat with 24 and 18} & \\
24 = q_2 (18) + r_2 & \\
24 = (1)(18) + 6 & \text{eq. 1} \\
\quad \text{Repeat with 18 and 6} & \\
18 = q_3 (6) + r_3 & \\
18 = (3)(6) + 0 & \\
\gcd(24, 138) \text{ is the last non-zero divisor} & \\
\gcd(24, 138) = 6 &
\end{array}
$$

Since $\gcd(24, 138) \mid 18$, the equation $24x + 138y = 18$ has infinitely many solutions.

First though, we solve the related equation, $24x + 138y = \gcd(24, 138)$.

(i.e., $24x + 138y = 6$).

$$
\begin{array}{l}
6 = 24 - (1)(18) \quad \text{(From eq.1)} \\
\qquad\qquad\qquad 18 = 138 - (5)(24) \quad \text{(From eq.2)} \\
\Rightarrow 6 = 24 - (1)(138 - (5)(24)) \\
\Rightarrow 6 = (6)(24) - (1)(138) \\
\text{i.e., } 24(6) + 138(-1) = 6
\end{array}
$$

Thus, $(x_0, y_0) = (6, -1)$ is a particular solution of the equation $24x + 138y = 6$.

We want the particular solution (x_p, y_p) to the equation $24x + 138y = \underbrace{18}_{c}$

To get (x_p, y_p), we multiply (x_0, y_0) by $\frac{c}{d} = \frac{18}{6} = 3$

Thus, $(x_p, y_p) = (18, -3)$ is a particular solution of the Diophantine equation $24x + 138y = 18$

To get the general solution, we must first find the homogeneous solution (x_h, y_h).

The homogeneous solution has the form:

$(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x_h, y_h) = \left(\frac{138}{6}t, -\frac{24}{6}t\right)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x_h, y_h) = (23t, -4t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

Our general solution $(x, y) = (x_p, y_p) + (x_h, y_h)$

$\Rightarrow (x, y) = (18, -3) + (23t, -4t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x, y) = (18 + 23t, -3 - 4t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$\text{i.e., } x = 18 + 23t; \quad y = -3 - 4t \quad \text{ for } t \in \mathbf{Z}$$

(c) $221x + 35y = 11$

First, use the Division Algorithm to find $\gcd(221, 35)$

$221 = q_1(35) + r_1$

$221 = (6)(35) + 11$ eq. 3

Repeat with 35 and 11

$35 = q_2(11) + r_2$

$35 = (3)(11) + 2$ eq. 2

Repeat with 11 and 2

$11 = q_3(2) + r_3$

$11 = (5)(2) + 1$ eq. 1

Repeat with 2 and 1

$2 = q_4(1) + r_4$

$2 = (2)(1) + 0$

$\gcd(221, 35)$ is the last non-zero divisor

$\gcd(221, 35) = 1$

Since $\gcd(221, 35) | 11$, the equation $221x + 35y = 11$ has infinitely many solutions.

First though, we solve the related equation, $221x + 35y = \gcd(221, 35)$.
(i.e., $221x + 35y = 1$).

$1 = 11 - (5)(2)$ (From eq. 1)

$2 = 35 - (3)(11)$ (From eq. 2)

$\Rightarrow 1 = 11 - (5)(35 - (3)(11))$

$\Rightarrow 1 = (-5)(35) + (16)(11)$

$11 = 221 - (6)(35)$ (From eq. 3)

$\Rightarrow 1 = (-5)(35) + (16)(221 - (6)(35))$

$\Rightarrow 1 = (16)(221) - (101)(35)$

i.e., $221(16) + 35(-101) = 1$

Thus, $(x_0, y_0) = (16, -101)$ is a particular solution of the equation $221x + 35y = 1$.

We want the particular solution (x_p, y_p) to the equation $221x + 35y = \underbrace{11}_{c}$

To get (x_p, y_p), we multiply (x_0, y_0) by $\frac{c}{d} = \frac{11}{1} = 11$

Thus, $(x_p, y_p) = (176, -1111)$ is a particular solution of the Diophantine equation $221x + 35y = 11$

To get the general solution, we must first find the homogeneous solution (x_h, y_h).

The homogeneous solution has the form:

$(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x_h, y_h) = \left(\frac{35}{1}t, -\frac{221}{1}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x_h, y_h) = (35t, -221t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

Our general solution $(x, y) = (x_p, y_p) + (x_h, y_h)$

$\Rightarrow (x, y) = (176, -1111) + (35t, -221t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x, y) = (176 + 35t, -1111 - 221t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

$$\text{i.e., } x = 176 + 35t; \quad y = -1111 - 221t \quad \text{for } t \in \mathbf{Z}$$

3. Determine all solutions in the *positive* integers of the following Diophantine equations:

(a) $18x + 5y = 48$

First, use the Euclidean Algorithm to find $\gcd(18, 5)$

$18 = q_1(5) + r_1$

$18 = (3)(5) + 3$ eq. 3

Repeat with 5 and 3

$5 = q_2(3) + r_2$

$5 = (1)(3) + 2$ eq. 2

Repeat with 3 and 2

$3 = q_3(2) + r_3$

$3 = (1)(2) + 1$ eq. 1

Repeat with 2 and 1

$2 = q_4(1) + r_4$

$2 = (2)(1) + 0$

$\gcd(18, 5)$ is the last non-zero divisor

$\gcd(18, 5) = 1$

Since $\gcd(18, 5) | 48$, the equation $18x + 5y = 48$ has infinitely many solutions.

First though, we solve the related equation, $18x + 5y = \gcd(18, 5)$.
(i.e., $18x + 5y = 1$).

$1 = 3 - (1)(2)$ (From eq. 1)

$2 = 5 - (1)(3)$ (From eq. 2)

$\Rightarrow 1 = 3 - (1)(5 - (1)(3))$

$\Rightarrow 1 = (-1)(5) + (2)(3)$

$3 = 18 - (3)(5)$ (From eq. 3)

$\Rightarrow 1 = (-1)(5) + (2)(18 - (3)(5))$

$\Rightarrow 1 = (2)(18) - (7)(5)$

i.e., $18(2) + 5(-7) = 1$

Thus, $(x_0, y_0) = (2, -7)$ is a particular solution of the equation $18x + 5y = 1$.

We want the particular solution (x_p, y_p) to the equation $18x + 5y = \underbrace{48}_{c}$

To get (x_p, y_p), we multiply (x_0, y_0) by $\frac{48}{d} = \frac{48}{1} = 48$

Thus, $(x_p, y_p) = (96, -336)$ is a particular solution of the Diophantine equation $18x + 5y = 48$

To get the general solution, we must first find the homogeneous solution (x_h, y_h).

The homogeneous solution has the form:

$(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x_h, y_h) = \left(\frac{5}{1}t, -\frac{18}{1}t\right)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x_h, y_h) = (5t, -18t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

Our general solution $(x, y) = (x_p, y_p) + (x_h, y_h)$

$\Rightarrow (x, y) = (96, -336) + (5t, -18t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x, y) = (96 + 5t, -336 - 18t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$\text{i.e., } x = 96 + 5t; \quad y = -336 - 18t \quad \text{for } t \in \mathbf{Z}$$

To find a solution in the positive integers, we need both $x > 0$ and $y > 0$.

$\Rightarrow x = 96 + 5t > 0$ and $y = -336 - 18t > 0$

$\Rightarrow 5t > -96$ and $-18t > 336$

$\Rightarrow t > -\frac{96}{5}$ and $t < -\frac{336}{18}$

$\Rightarrow t > -19.2$ and $t < -18.667$

$\Rightarrow -19.2 < t < -18.667$

$\Rightarrow t = -19$

$\Rightarrow x = 96 + 5(-19)$ and $y = -336 - 18(-19)$

$\Rightarrow x = 1$ and $y = 6$ yield the only solution in the positive integers.

(b) $54x + 21y = 906$

First, use the Euclidean Algorithm to find $\gcd(54, 21)$

$54 = q_1(21) + r_1$

$54 = (2)(21) + 12$ eq. 3

Repeat with 21 and 12

$21 = q_2(12) + r_2$
$21 = (1)(12) + 9$ eq. 2

Repeat with 12 and 9

$12 = q_3(9) + r_3$

$12 = (1)(9) + 3$ eq. 1

Repeat with 9 and 3

$9 = q_4(3) + r_4$

$9 = (3)(3) + 0$

$\gcd(54, 21)$ is the last non-zero divisor

$\gcd(54, 21) = 3$

Since $\gcd(54, 21) \,|906$, the equation $54x+21y = 906$ has infinitely many solutions.

First though, we solve the related equation, $54x + 21y = \gcd(54, 21)$.
(i.e., $54x + 21y = 3$).

$3 = 12 - (1)(9)$ (From eq.1)

$9 = 21 - (1)(12)$ (From eq. 2)

$\Rightarrow 3 = 12 - (1)(21 - (1)(12))$

$\Rightarrow 3 = (-1)(21) + (2)(12)$

$12 = 54 - (2)(21)$ (From eq.3)

$\Rightarrow 3 = (-1)(21) + (2)(54 - (2)(21))$

$\Rightarrow 3 = (2)(54) - (5)(21)$

i.e., $54(2) + 21(-5) = 3$

Thus, $(x_0, y_0) = (2, -5)$ is a particular solution of the equation $54x + 21y = 3$.

We want the particular solution (x_p, y_p) to the equation $54x + 21y = \underbrace{906}_{c}$

To get (x_p, y_p), we multiply (x_0, y_0) by $\frac{c}{d} = \frac{906}{3} = 302$

Thus, $(x_p, y_p) = (604, -1510)$ is a particular solution of the Diophantine equation $54x + 21y = 906$

To get the general solution, we must first find the homogeneous solution (x_h, y_h).

The homogeneous solution has the form:

$(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x_h, y_h) = \left(\frac{21}{3}t, -\frac{54}{3}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x_h, y_h) = (7t, -18t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

Our general solution $(x, y) = (x_p, y_p) + (x_h, y_h)$

$\Rightarrow (x, y) = (604, -1510) + (7t, -18t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x, y) = (604 + 7t, -1510 - 18t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

$$\text{i.e., } x = 604 + 7t; \quad y = -1510 - 18t \quad \text{ for } t \in \mathbf{Z}$$

To find a solution in the positive integers, we need both $x > 0$ and $y > 0$.

$\Rightarrow x = 604 + 7t > 0$ and $y = -1510 - 18t > 0$

$\Rightarrow 7t > -604$ and $-18t > 1510$

$\Rightarrow t > -\frac{604}{7}$ and $t < -\frac{1510}{18}$

$\Rightarrow t > -86.286$ and $t < -83.89$

$\Rightarrow -86 \leq t \leq 84$

$\boxed{t = -86}$

$x = 604 + 7t = 604 + 7(-86) = 2; \quad y = -1510 - 18t = -1510 - 18(-86) = 38$

$\boxed{t = -85}$

$x = 604 + 7t = 604 + 7(-85) = 9; \quad y = -1510 - 18t = -1510 - 18(-85) = 20$

$\boxed{t = -84}$

$x = 604 + 7t = 604 + 7(-84) = 16; \quad y = -1510 - 18t = -1510 - 18(-84) = 2$

$\boxed{(2, 38);\ (9, 20);\ (16, 2) \text{ are the only solutions in the positive integers}}$

(c) $123x + 360y = 99$

First, use the Euclidean Algorithm to find $\gcd(123, 360)$

$360 = q_1 (123) + r_1$

$360 = (2)(123) + 114$ eq. 4

Repeat with 123 and 114

$123 = q_2 (114) + r_2$

$123 = (1)(114) + 9$ eq. 3

Repeat with 114 and 9

$114 = q_3 (9) + r_3$

$114 = (12)(9) + 6$ eq. 2

Repeat with 9 and 6

$9 = q_4 (6) + r_4$

$9 = (1)(6) + 3$ eq.1

Repeat with 6 and 3

$6 = q_5 (3) + r_5$

$6 = (3)(3) + 0$

$\gcd(123, 360)$ is the last non-zero divisor

$\gcd(123, 360) = 3$

Since $\gcd(123, 360) | 99$, the equation $123x + 360y = 99$ has infinitely many solutions.

First, we solve the related equation $123x + 360y = \gcd(123, 360)$. (i.e., $123x + 360y = 3$).

$3 = 9 - (1)(6)$ (From eq. 1)

$6 = 114 - (12)(9)$ (From eq.2)

$\Rightarrow 3 = 9 - (1)(114 - (12)(9))$

$\Rightarrow 3 = (-1)(114) + (13)(9)$

$9 = 123 - (1)(114)$ (From eq. 3)

$\Rightarrow 3 = (-1)(114) + (13)(123 - (1)(114))$

$\Rightarrow 3 = (13)(123) - (14)(114)$

$114 = 360 - (2)(123)$ (From eq. 4)

$\Rightarrow 3 = (13)(123) - (14)(360 - (2)(123))$

$\Rightarrow 3 = (41)(123) - (14)(360)$

i.e., $123(41) + 360(-14) = 3$

Thus, $(x_0, y_0) = (41, -14)$ is a particular solution of the equation $123x+360y = 3$.

We want the particular solution (x_p, y_p) to the equation $123x + 360y = \underbrace{99}_{c}$

To get (x_p, y_p), we multiply (x_0, y_0) by $\frac{c}{d} = \frac{99}{3} = 33$

Thus, $(x_p, y_p) = (1353, -462)$ is a particular solution of the Diophantine equation $123x + 360y = 99$

To get the general solution, we must first find the homogeneous solution (x_h, y_h).

The homogeneous solution has the form:

$(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x_h, y_h) = \left(\frac{360}{3}t, -\frac{123}{3}t\right)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x_h, y_h) = (120t, -41t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

Our general solution $(x, y) = (x_p, y_p) + (x_h, y_h)$

$\Rightarrow (x, y) = (1353, -462) + (120t, -41t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

i.e., $(x, y) = (1353 + 120t, -462 - 41t)$ for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$\text{i.e., } x = 1353 + 120t; \quad y = -462 - 41t \quad \text{ for } t \in \mathbf{Z}$$

To find a solution in the positive integers, we need both $x > 0$ and $y > 0$.

$\Rightarrow x = 1353 + 120t > 0$ and $y = -462 - 41t > 0$

$\Rightarrow 120t > -1353$ and $-41t > 462$

$\Rightarrow t > -\frac{1353}{120}$ and $t < -\frac{462}{41}$

$\Rightarrow t > -11.275$ and $t < -11.268$

$\Rightarrow -11.275 < t < -11.268$

Since no *integer* value of t satisfies this inequality, **there are NO solutions in the positive numbers.**

(d) $158x - 57y = 7$

First, use the Division Algorithm to find $\gcd(158, -57) = \gcd(158, 57)$

$158 = q_1(57) + r_1$

$158 = (2)(57) + 44$ eq. 6

Repeat with 57 and 44

$57 = q_2(44) + r_2$

$57 = (1)(44) + 13$ eq. 5

Repeat with 44 and 13

$44 = q_3(13) + r_3$

$44 = (3)(13) + 5$ eq. 4

Repeat with 13 and 5

$13 = q_4(5) + r_4$

$13 = (2)(5) + 3$ eq. 3

Repeat with 5 and 3

$5 = q_5(3) + r_5$

$5 = (1)(3) + 2$ eq. 2

Repeat with 3 and 2

$3 = q_6(2) + r_6$

$3 = (1)(2) + 1$ eq. 1

Repeat with 2 and 1

$2 = q_7(1) + r_7$

$2 = (2)(1) + 0$

$\gcd(158, 57)$ is the last non-zero divisor

$\gcd(158, 57) = 1$

Since $\gcd(158, 57) | 7$, the equation $158x - 57y = 7$ has infinitely many solutions.

First though, we solve the related equation, $158x - 57y = \gcd(158, 57)$.
(i.e., $158x - 57y = 1$).

$$1 = 3 - (1)(2) \quad \text{(From eq. 1)}$$

$$2 = 5 - (1)(3) \quad \text{(From eq. 2)}$$

$$\Rightarrow 1 = 3 - (1)(5 - (1)(3))$$

$$\Rightarrow 1 = (-1)(5) + (2)(3)$$

$$3 = 13 - (2)(5) \quad \text{(From eq. 3)}$$

$$\Rightarrow 1 = (-1)(5) + (2)(13 - (2)(5))$$

$$\Rightarrow 1 = (2)(13) - (5)(5)$$

$$5 = 44 - (3)(13) \quad \text{(From eq. 4)}$$

$$\Rightarrow 1 = (2)(13) - (5)(44 - (3)(13))$$

$$\Rightarrow 1 = (-5)(44) + (17)(13)$$

$$13 = 57 - (1)(44) \quad \text{(From eq. 5)}$$

$$\Rightarrow 1 = (17)(57 - (1)(44)) - (5)(44)$$

$$\Rightarrow 1 = (17)(57) - (22)(44)$$

$$44 = 158 - (2)(57) \quad \text{(From eq. 6)}$$

$$\Rightarrow 1 = (17)(57) - (22)(158 - (2)(57))$$

$$\Rightarrow 1 = 158(-22) + 57(61) \quad \text{i.e., } 158(-22) - 57(-61) = 1$$

Thus, $(x_0, y_0) = (-22, -61)$ is a particular solution of the equation $158x - 57y = 1$.

We want the particular solution (x_p, y_p) to the equation $158x - 57y = \underbrace{7}_{c}$

To get (x_p, y_p), we multiply (x_0, y_0) by $\frac{c}{d} = \frac{7}{1} = 7$

Thus, $(x_p, y_p) = (-154, -427)$ is a particular solution of the Diophantine equation $158x - 57y = 7$

To get the general solution, we must first find the homogeneous solution (x_h, y_h).

The homogeneous solution has the form:

$(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x_h, y_h) = \left(-\frac{57}{1}t, -\frac{158}{1}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x_h, y_h) = (-57t, -158t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

Our general solution $(x, y) = (x_p, y_p) + (x_h, y_h)$

$\Rightarrow (x, y) = (-154, -427) + (-57t, -158t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x, y) = (-154 - 57t, -427 - 158t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

$$x = -154 - 57t; \quad y = -427 - 158t \quad \text{for } t \in \mathbf{Z}$$

To find a solution in the positive integers, we need both $x > 0$ and $y > 0$.

$\Rightarrow x = -154 - 57t > 0$ and $y = -427 - 158t > 0$

$\Rightarrow -57t > 154$ and $-158t > 427$

$\Rightarrow t < -\frac{154}{57}$ and $t < -\frac{427}{158}$

$\Rightarrow t < -2.7018$ and $t < -2.7025$

$\Rightarrow t < -2.7025$

The solutions (in the positive integers) of the equation $158x - 57y = 7$ is given by

$$\text{i.e., } x = -154 - 57t; \quad y = -427 - 158t \quad \text{for } t \in \{\ldots, -6, -5, -4, -3\}$$

4. If a and b are relatively prime integers, prove that the Diophantine equation $ax-by=c$ has infinitely many solutions in the positive integers.

Proof. Let the hypotheses be given. Since $\gcd(a,b)=1$, it follows that $\gcd(a,-b)=1$, and hence, $\gcd(a,-b)\,|c$.

Hence, we are guaranteed that there are infinitely many solutions of the equation $ax-by=c$.

Let (x_0,y_0) be *any* particular solution of the equation $ax-by=c$.

The general solution of the equation $ax-by=c$ is the same as the general solution of the equation $ax+(-b)\,y=c$ which is given by:

$$x=x_0+\left(\frac{-b}{d}\right)t;\quad y=y_0-\left(\frac{a}{d}\right)t\quad \text{for } t\in\mathbf{Z},$$

where $d=\gcd(a,b)$.

Since $d=\gcd(a,b)=1$, we have:

$$x=x_0-bt;\quad y=y_0-at\quad \text{for } t\in\mathbf{Z},$$

To find a solution in the positive integers, we need both $x>0$ and $y>0$.

$\Rightarrow x=x_0-bt>0$ and $y=y_0-at>0$

$\Rightarrow -bt>-x_0$ and $-at>-y_0$

$t<\frac{x_0}{b}$ and $t<\frac{y_0}{a}$

(Note: Since, a and b are both positive, $-a$ and $-b$ are both negative. Hence, when we divide by either $-a$ or $-b$, the inequality is reversed.)

If we define m to be the greatest integer less than $\min\left\{\frac{x_0}{b},\frac{y_0}{a}\right\}$, the inequality $t\le m$ satisfies both inequalities $t<\frac{x_0}{b}$ and $t<\frac{y_0}{a}$.

Hence,

$$x=x_0-bt;\quad y=y_0-at\quad \text{for } t\in\{\ldots,m-2,m-1,m\}$$

generates solutions (in the positive integers) of the equation $ax-by=c$. ■

6. ~

a. A man has \$4.55 in change composed entirely of dimes and quarters. What are the maximum and minimum number of coins that he can have? Is it possible for the number of dimes to equal the number of quarters?

Let x be the number of quarters and y be the number of dimes. Then the situation is modeled by the equation

$$25x + 10y = 455.$$

We are looking for solutions in the non-negative integers. (We can't have a negative number of quarters and/or dimes.)
First, use the Division Algorithm to find $\gcd(25, 10)$.

$25 = q_1 (10) + r_1$
$25 = (2)(10) + 5$ eq. 1
Repeat with 10 and 5
$10 = q_2 (5) + r_2$
$10 = (2)(5) + 0$
$\gcd(25, 10)$ is the last non-zero divisor
$\gcd(25, 10) = 5$

Since $\gcd(25, 10) | 455$, the equation $25x+10y = 455$ has infinitely many solutions.

First though, we solve the related equation, $25x + 10y = \gcd(25, 10)$.
(i.e., $25x + 10y = 5$).

From eq. 1, we have: $5 = 25 - (2)(10)$
i.e., $25(1) + 10(-2) = 5$

Next, we multiply both sides of this equation by 91 ($91 \cdot 5 = 455$).

$\Rightarrow 91[25(1) + 10(-2)] = 91(5)$

$\Rightarrow 25(91) + 10(-182) = 455$

Thus, $(x_0, y_0) = (91, -182)$ is a particular solution of the equation

$25x + 10y = 455$.

To find the general solution, recall that the general solution of the equation $ax + by = c$ is given by:

$x = x_0 + \left(\frac{b}{d}\right) t; \quad y = y_0 - \left(\frac{a}{d}\right) t$ for $t \in \mathbf{Z}$, where $d = \gcd(a, b)$.

Thus, the general solution of the equation $25x + 10y = 455$ is given by

$$x = 91 + \left(\frac{10}{5}\right) t; \quad y = -182 - \left(\frac{25}{5}\right) t \quad \text{for } t \in \mathbf{Z}$$

$$\text{i.e., } x = 91 + 2t; \quad y = -182 - 5t \quad \text{for } t \in \mathbf{Z}$$

To find a solution in the non-negative integers, we need both $x \geq 0$ and $y \geq 0$.

$\Rightarrow x = 91 + 2t \geq 0$ and $y = -182 - 5t \geq 0$

$\Rightarrow 2t \geq -91$ and $-5t \geq 182$

$\Rightarrow t \geq -\frac{91}{2}$ and $t \leq -\frac{182}{5}$

$\Rightarrow t \geq -45.5$ and $t \leq -36.4$

$\Rightarrow -45 \leq t \leq -37$

The solutions (in the non-negative integers) of the equation $25x + 10y = 455$ is given by

$$\text{i.e., } x = 91 + 2t; \quad y = -182 - 5t \quad \text{for } -45 \leq t \leq -37$$

We have the maximum number of coins when the number of dimes is as large as possible. This occurs when $t = -45$.

$\Rightarrow x = 91 + 2t = 91 + 2(-45) = 1$

$\Rightarrow y = -182 - 5t = -182 - 5(-45) = 43$

We have the maximum number of coins when the number of quarters is 1 and the number of dimes is 43 (44 total coins).

We have the minimum number of coins when the number of quarters is as large as possible. This occurs when $t = -37$.

$\Rightarrow x = 91 + 2t = 91 + 2(-37) = 17$

$\Rightarrow y = -182 - 5t = -182 - 5(-37) = 3$

We have the minimum number of coins when the number of quarters is 17 and the number of dimes is 3 (20 total coins).

Can the number of coins be equal? If so, then $x = y$.

$\Rightarrow x = 91 + 2t = -182 - 5t = y$

$\Rightarrow 91 + 2t = -182 - 5t$

$\Rightarrow 7t = -273$

$\Rightarrow t = -39$

$\Rightarrow x = 91 + 2t = 91 + 2(-39) = 13$

$\Rightarrow y = -182 - 5t = -182 - 5(-39) = 13$

We can have 13 quarters and 13 dimes, and have a total of $4.55.

b. The neighborhood theater charges \$1.80 for adult admissions and \$0.75 for children. On a particular evening, the total receipts were \$90.00. Assuming that more adults than children were present, how many children attended?

Let x be the number of adults and y be the number of children. Then the situation is modeled by the equation

$$180x + 75y = 9000.$$

We are looking for solutions in the non-negative integers. (We can't have a negative number of adults and/or children.)

First, use the Division Algorithm to find $\gcd(180, 75)$.

$180 = q_1 (75) + r_1$
$180 = (2)(75) + 30$ eq. 2
Repeat with 75 and 30
$75 = q_2 (30) + r_2$
$75 = (2)(30) + 15$ eq. 1
Repeat with 30 and 15
$30 = q_3 (15) + r_3$
$30 = (2)(15) + 0$
$\gcd(180, 75)$ is the last non-zero divisor
$\gcd(180, 75) = 15$

Since $\gcd(180, 75) | 9000$, the equation $180x + 75y = 9000$ has infinitely many solutions.

First though, we solve the related equation, $180x + 75y = \gcd(180, 75)$.
(i.e., $180x + 75y = 15$).

$15 = 75 - (2)(30)$ (From eq. 1)
$30 = 180 - (2)(75)$ (From eq. 2)
$\Rightarrow 15 = 75 - (2)(180 - (2)(75))$
$\Rightarrow 15 = (5)(75) - (2)(180)$
i.e., $180(-2) + 75(5) = 15$

Next, we multiply both sides of this equation by 600 ($600 \cdot 15 = 9000$).

$\Rightarrow 600 [180(-2) + 75(5)] = 600(15)$

$\Rightarrow 180(-1200) + 75(3000) = 9000$

Thus, $(x_0, y_0) = (-1200, 3000)$ is a particular solution of the equation

$180x + 75y = 9000.$

To find the general solution, recall that the general solution of the equation

$ax + by = c$ is given by:

$$x = x_0 + \left(\frac{b}{d}\right)t; \quad y = y_0 - \left(\frac{a}{d}\right)t \quad \text{for } t \in \mathbf{Z}, \text{ where } d = \gcd(a, b).$$

Thus, the general solution of the equation $180x + 75y = 9000$ is given by

$$x = -1200 + \left(\frac{75}{15}\right)t; \quad y = 3000 - \left(\frac{180}{15}\right)t \quad \text{for } t \in \mathbf{Z}$$

$$\text{i.e., } x = -1200 + 5t; \quad y = 3000 - 12t \quad \text{for } t \in \mathbf{Z}$$

To find a solution in the non-negative integers, we need both $x \geq 0$ and $y \geq 0$.

$\Rightarrow x = -1200 + 5t \geq 0$ and $y = 3000 - 12t \geq 0$

$\Rightarrow 5t \geq 1200$ and $-12t \geq -3000$

$\Rightarrow t \geq \frac{1200}{5}$ and $t \leq \frac{3000}{12}$

$\Rightarrow t \geq 240$ and $t \leq 250$

$\Rightarrow 240 \leq t \leq 250$

Since more adults than children attended, we also want $x > y$.

$\Rightarrow 3000 - 12t < -1200 + 5t$

$\Rightarrow 4200 < 17t$

$\Rightarrow \frac{4200}{17} < t$

$\Rightarrow 247.06 < t$

$\Rightarrow t = 248, 249, 250$

The solutions that we seek are given by

$$\text{i.e., } x = -1200 + 5t; \quad y = 3000 - 12t \quad \text{for } 248 \leq t \leq 250$$

Here are the possibilities:

$x = 40 \qquad y = 24 \qquad (t = 248)$

$x = 45 \qquad y = 12 \qquad (t = 249)$

$x = 50 \qquad y = 0 \qquad (t = 250)$

c. A certain number of sixes and nines is added to give a sum of 126; if the number of sixes and nines is interchanged, the new sum is 114. How many of each were there originally?

This problem yields the set of equations:

$$9x + 6y = 126 \qquad (x = \# \text{ of nines})$$

$$6x + 9y = 114 \qquad (x = \# \text{ of sixes})$$

The same x-value and the same y-value work for both equations. (i.e., these equations must be true *simultaneously*.)

Since these equations don't represent the same line or parallel lines, the system can have only one solution.

If that solution is not "a solution in the integers," we're out of luck. There's no solution.

Solving this *system* of equations yields $x = 10$; $y = 6$.

$$9\,(10) + 6\,(6) = 126 \qquad (x = \# \text{ of nines})$$

$$6\,(10) + 9\,(6) = 114 \qquad (x = \# \text{ of sixes})$$

Originally, there were 10 nines and 6 sixes.

7. When Mr. Smith cashed a check at his bank, the teller mistook the number of cents for the number of dollars and vice versa. Unaware of this, Mr. Smith spent 68 cents and then noticed, to his surprised, that he had twice the amount of the original check. Determine the smallest value for which the check could have been written.

   Let $x =$ the number of dollars for which the check was written.

   Let $y =$ the number of cents for which the check was written.

   Then $100x + y$ is the amount (in cents) for which the check was written.

   Also, $100y + x$ is the amount (in cents) that the teller *thought* that the check was written for, and consequently:

   $100y + x$ is the amount (in cents) that the teller gave to Mr. Smith.

   We are also told that after Mr. Smith spent 68 cents, he had twice the amount (in cents) for which the check was written.

   OBSERVE:

   Mr. Smith was given $100y + x$ (in cents) by the teller.

   After Mr. Smith spent 68 cents, he had $100y + x - 68$ (in cents).

   We are told that this amount is twice the amount for which the check was written. (i.e., $100y + x - 68 = 2\,(100x + y)$)

   Simplifying, we have:

   $$100y + x - 68 = 200x + 2y$$

   $$\Rightarrow \qquad -199x + 98y = 68$$

   Thus, we have a Diophantine equation with

   $a = -199$

   $b = 98$

   $c = 68$

   $d = \gcd(a, b) = \gcd(-199, 98) = 1$

   Since $d|c$ (i.e., $1|68$), the equation has a solution.

Using the Division Algorithm to find $\gcd(-199, 98)$, we have:

$-199 = q_1(98) + r_1$

$-199 = (-3)(98) + 95$ eq. 4

Repeat with 98 and 95

$98 = q_2(95) + r_2$

$98 = (1)(95) + 3$ eq. 3

Repeat with 95 and 3

$95 = q_3(3) + r_3$

$95 = (31)(3) + 2$ eq. 2

Repeat with 3 and 2

$3 = q_4(2) + r_4$

$3 = (1)(2) + 1$ eq. 1

Repeat with 2 and 1

$2 = q_5(1) + r_5$

$2 = 2(1)$

$\gcd(-199, 98)$ is the last non-zero divisor

$d = \gcd(-199, 98) = 1$

Since $\gcd(-199, 98) \mid 68$, the equation $-199x + 98y = 68$ has infinitely many solutions.

First though, we solve the related equation, $-199x + 98y = \gcd(-199, 98)$.
(i.e., $-199x + 98y = 1$).

$1 = 3 - (1)(2)$ (From eq. 1)

$2 = 95 - (31)(3)$ (From eq. 2)

$\Rightarrow 1 = 3 - (1)[95 - (31)(3)]$

$\Rightarrow 1 = (32)(3) - 95$

$3 = 98 - (1)(95)$ (From eq. 3)

$\Rightarrow 1 = (32)[98 - (1)(95)] - 95$

$\Rightarrow 1 = (32)(98) - (33)(95)$

$95 = -199 + (3)(98)$ (From eq. 4)

$\Rightarrow 1 = (32)(98) - (33)[-199 + (3)(98)]$

$\Rightarrow 1 = (-67)(98) - (33)(-199)$

Re-express in the exact form $ax + by = \gcd(a, b)$

$\Rightarrow \quad -199(-33) + 98(-67) = 1$

Thus, $(x_0, y_0) = (-33, -67)$ is a particular solution of the equation $-199x + 98y = 1$.

We want the particular solution (x_p, y_p) to the equation $-199x + 98y = \underbrace{68}_{c}$

To get (x_p, y_p), we multiply (x_0, y_0) by $\frac{c}{d} = \frac{68}{1} = 68$

Thus, $(x_p, y_p) = (-2244, -4556)$ is a particular solution of the Diophantine equation $-199x + 98y = 68$

To get the general solution, we must first find the homogeneous solution (x_h, y_h).

The homogeneous solution has the form:

$(x_h, y_h) = \left(\frac{b}{d}t, -\frac{a}{d}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x_h, y_h) = \left(\frac{98}{1}t, -\frac{-199}{1}t\right)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x_h, y_h) = (98t, 199t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

Our general solution $(x, y) = (x_p, y_p) + (x_h, y_h)$

$\Rightarrow (x, y) = (-2244, -4556) + (98t, 199t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

i.e., $(x, y) = (-2244 + 98t, -4556 + 199t)$ for $t = 0, \pm1, \pm2, \pm3, \ldots$

To find the smallest value for which the check could have been written, realize that both x and y (the number of dollars and cents. respectively) must be positive.

Therefore, $-2244 + 98t > 0$ and $-4556 + 199t > 0$

$\Rightarrow 98t > 2244$ and $199t > 4556$

$\Rightarrow t > \frac{2244}{98} = 22.89$ and $t > \frac{4556}{199}$

i.e., $t > 22.89$ and $t > 22.89$

Hence, $t \geq 23$ yields positive solutions for x and y.

Of these solutions, the smallest value for which the check could have been written will have the smallest possible positive x-value (the number of dollars).

This occurs when $t = 23$.

$x = -2244 + 98\,(23) = 10$

$y = -4556 + 199\,(23) = 21$

The check was written for \$10.21.

Check: $\$21.10 - \$0.68 = \$20.42 = 2\,(\$10.21)$