MTH 4441 Homework #1 - Binary Operators - Solutions

Fall 2017

Pat Rossi Name ______________________

In exercises 1-7, determine whether:

i) $*$ is a binary operation on S

ii) $*$ is commutative

iii) $*$ is associative

Where appropriate, we may use the facts that addition and multiplication of integers and/or real numbers is both commutative and associative.

You may also make reference to the fact that the operations of addition and multiplication are "closed" on the integers and real numbers.

1. $S = \mathbb{Z}$ and $a * b = 2a + 2b$

   i) $*$ IS a binary operation on $\mathbb{Z}$, since it assigns the ordered pair of integers (a, b) to $2a + 2b$, which is also an integer, since addition and multiplication is closed on $\mathbb{Z}$.

   ii) $*$ is commutative, since $a * b = 2a + 2b = 2b + 2a = b * a$

   i.e., $a * b = b * a$

   iii) $*$ is NOT associative!

   $a * (b * c) = 2a + 2(b * c) = 2a + 2(2b + 2c) = 2a + 4b + 4c$

   however:

   $(a * b) * c = 2(a * b) + 2c = 2(2a + 2b) + 2c = 4a + 4b + 2c$

   Thus, $a * (b * c) \neq (a * b) * c$

   Alternatively:

   $a * (b * c) = a * (2b + 2c) = 2a + 2(2b + 2c) = 2a + 4b + 4c$

   however:

   $(a * b) * c = (2a + 2b) * c = 2(2a + 2b) + 2c = 4a + 4b + 2c$

   Thus, $a * (b * c) \neq (a * b) * c$

2. $S = \mathbb{Z}$ and $a * b = a + 2b$

i) $*$ IS a binary operation on $\mathbb{Z}$, since it assigns the ordered pair of integers (a, b) to $a + 2b$, which is also an integer, since addition and multiplication is closed on $\mathbb{Z}$.

ii) $*$ is NOTcommutative, since $a * b = a + 2b \neq b + 2a = b * a$

i.e., $a * b \neq b * a$

iii) $*$ is NOT associative!

$$a * (b * c) = a + 2\,(b * c) = a + 2\,(b + 2c) = a + 2b + 4c$$

however:

$$(a * b) * c = (a * b) + 2c = (a + 2b) + 2c = a + 2b + 2c$$

Thus, $a * (b * c) \neq (a * b) * c$

Alternatively:

$$a * (b * c) = a * (b + 2c) = a + 2\,(b + 2c) = a + 2b + 4c$$

however:

$$(a * b) * c = (a + 2b) * c = (a + 2b) + 2c = a + 2b + 2c$$

Thus, $a * (b * c) \neq (a * b) * c$

3. $S = \mathbb{Z}$ and $a * b = 2a - 2b$

i) $*$ IS a binary operation on $\mathbb{Z}$, since it assigns the ordered pair of integers (a, b) to $2a - 2b$, which is also an integer, since addition and multiplication is closed on $\mathbb{Z}$.

ii) $*$ is NOTcommutative, since $a * b = 2a - 2b \neq 2b - 2a = b * a$

i.e., $a * b \neq b * a$

To cite a counterexample: $2 * 3 = 2(2) - 2(3) = -2$

but $3 * 2 = 2(3) - 2(2) = 2$

Hence, $2 * 3 \neq 3 * 2$

iii) $*$ is NOT associative!

$a * (b * c) = 2a - 2(b * c) = 2a - 2(2b - 2c) = 2a - 4b + 4c$

however:

$(a * b) * c = 2(a * b) - 2c = 2(2a - 2b) - 2c = 4a - 4b - 2c$

Thus, $a * (b * c) \neq (a * b) * c$

Alternatively:

$a * (b * c) = 2a * (2b - 2c) = 2a - 2(2b - 2c) = 2a - 4b + 4c$

however:

$(a * b) * c = (2a - 2b) * c = 2(2a - 2b) - 2c = 4a - 4b - 2c$

Thus, $a * (b * c) \neq (a * b) * c$

4. $S = \mathbb{R}$ and $a * b = \frac{a}{b}$

i) $*$ is NOT a binary operation on $\mathbb{R}$, since it does not assign the ordered pair of real numbers $(a, 0)$ to any element of $\mathbb{R}$. (By definition of $*$, $a * 0 = \frac{a}{0}$, which is not defined. Thus $*$ is not closed on $\mathbb{R}$.)

ii) $*$ is NOTcommutative, since $a * b = \frac{a}{b} \neq \frac{b}{a} = b * a$

i.e., $a * b \neq b * a$ for all real numbers $a, b \neq \pm 1$

To cite a counterexample: $4 * 2 = \frac{4}{2} = 2$

but $2 * 4 = \frac{2}{4} = \frac{1}{2}$

Hence, $4 * 2 \neq 2 * 4$

iii) $*$ is NOT associative!

$$a * (b * c) = \frac{a}{b*c} = \frac{a}{\left(\frac{b}{c}\right)} = a \cdot \frac{1}{\left(\frac{b}{c}\right)} = a \cdot \frac{c}{b} = \frac{ac}{b}$$

however:

$$(a * b) * c = \frac{a*b}{c} = \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{b}\frac{1}{c} = \frac{a}{bc}$$

Thus, $a * (b * c) \neq (a * b) * c$

Alternatively:

$$a * (b * c) = a * \left(\frac{b}{c}\right) = \frac{a}{\left(\frac{b}{c}\right)} = a\frac{c}{b} = \frac{ac}{b}$$

however:

$$(a * b) * c = \frac{(a*b)}{c} = \frac{\left(\frac{a}{b}\right)}{c} = \left(\frac{a}{b}\right)\frac{1}{c} = \frac{a}{bc}$$

Thus, $a * (b * c) \neq (a * b) * c$

5. $S = \mathbb{R}^+$ and $a * b = \frac{a}{b}$

i) $*$ IS a binary operation on $\mathbb{R}^+$, since it assigns the ordered pair of positive real numbers (a, b) to the element $\frac{a}{b}$ which is also element of $\mathbb{R}^+$ for all $a, b \in \mathbb{R}^+$.

ii) $*$ is NOTcommutative, since $a * b = \frac{a}{b} \neq \frac{b}{a} = b * a$

i.e., $a * b \neq b * a$ for all real numbers $a, b \neq 1$

To cite a counterexample: $4 * 2 = \frac{4}{2} = 2$

but $2 * 4 = \frac{2}{4} = \frac{1}{2}$

Hence, $4 * 2 \neq 2 * 4$

iii) $*$ is NOT associative!

$a * (b * c) = \frac{a}{b*c} = \frac{a}{\left(\frac{b}{c}\right)} = a \cdot \frac{1}{\left(\frac{b}{c}\right)} = a \cdot \frac{c}{b} = \frac{ac}{b}$

however:

$(a * b) * c = \frac{a*b}{c} = \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{b}\frac{1}{c} = \frac{a}{bc}$

Thus, $a * (b * c) \neq (a * b) * c$

Alternatively:

$a * (b * c) = a * \left(\frac{b}{c}\right) = \frac{a}{\left(\frac{b}{c}\right)} = a\frac{c}{b} = \frac{ac}{b}$

however:

$(a * b) * c = \frac{(a*b)}{c} = \frac{\left(\frac{a}{b}\right)}{c} = \left(\frac{a}{b}\right)\frac{1}{c} = \frac{a}{bc}$

Thus, $a * (b * c) \neq (a * b) * c$

6. $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \; a, b, c, d \in \mathbb{R} \right\}$ and $A * B = A + B$ (The usual matrix addition)

i) $*$ IS a binary operation on S, since it assigns the ordered pair (A, B) of matrices with real entries to $A + B$, which is also a 2×2 matrix with real entries.

ii) $*$ is commutative. Given $A, B \in S$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, we have:

$$A * B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} e+a & f+b \\ g+c & h+d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = B * A$$

i.e., $A * B = B * A$

iii) $*$ IS associative! Given $A, B, C \in S$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, and $C = \begin{bmatrix} i & j \\ k & l \end{bmatrix}$ we have:

$$A * (B * C) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} i & j \\ k & l \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} (e+i) & (f+j) \\ (g+k) & (h+l) \end{bmatrix}$$

$$= \begin{bmatrix} (a+e+i) & (b+f+j) \\ (c+g+k) & (d+h+l) \end{bmatrix}$$

and:

$$(A * B) * C = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) + \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix} + \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$$= \begin{bmatrix} (a+e+i) & (b+f+j) \\ (c+g+k) & (d+h+l) \end{bmatrix}$$

Thus, $A * (B * C) = (A * B) * C$

7. $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ a, b, c, d \in \mathbb{R} \right\}$ and $A * B = AB$ (The usual matrix multiplication)

i) $*$ IS a binary operation on S, since it assigns the ordered pair (A, B) of matrices with real entries to AB, which is also a 2×2 matrix with real entries.

ii) $*$ is NOT commutative. Take, for example, $A, B \in S$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. we have:

$$A * B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

However:

$$B * A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Hence, $A * B \neq B * A$

iii) $*$ IS associative! Given $A, B, C \in S$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, and $C = \begin{bmatrix} i & j \\ k & l \end{bmatrix}$we have:

$$A * (B * C) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} (ie + fk) & (ej + fl) \\ (gi + hk) & (gj + hl) \end{bmatrix}$$

$$= \begin{bmatrix} (aie + afk + big + bhk) & (aej + afl + bgj + bhl) \\ (cei + cfk + dgi + dhk) & (cej + cfl + dgj + dhl) \end{bmatrix}$$

and:

$$(A * B) * C = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$$= \begin{bmatrix} (afk + bhk + aei + bgi) & (aej + bgj + afl + bhl) \\ (cfk + dhk + cei + dgi) & (cej + dgj + cfl + dhl) \end{bmatrix}$$

$$= \begin{bmatrix} (aie + afk + bgi + bhk) & (aej + afl + bgj + bhl) \\ (cei + cfk + dgi + dhk) & (cej + cfl + dgj + dhl) \end{bmatrix}$$

Thus, $A * (B * C) = (A * B) * C$