MTH 4441 Homework - Groups and Group Axioms - Solutions

Fall 2017

Pat Rossi Name ______________________

Decide whether each of the given sets is a group with respect to the given operation. If it is NOT a group, state one of the group axioms that fails to hold.

1. The set $\mathbb{Z}^+$ of all positive integers with operation addition.

   $(\mathbb{Z}^+, +)$ is **NOT a group.** The additive inverse of each element $n \in \mathbb{Z}^+$ is the *negative* integer $-n$, which is NOT an element of $\mathbb{Z}^+$.

2. The set $\mathbb{Z}^+$ of all positive integers with operation multiplication.

   $(\mathbb{Z}^+, \cdot)$ is **NOT a group.** The multiplicative inverse of each element $n \in \mathbb{Z}^+$ is the rational number $\frac{1}{n}$, which is NOT an element of $\mathbb{Z}^+$ for $n > 1$.

3. The set $\mathbb{Q}$ of all rational numbers with operation addition.

   $(\mathbb{Q}, +)$ **IS a group.**

   The operation $+$ is closed on $\mathbb{Q}$, since the sum of two rational numbers is also a rational number.

   $0 \in \mathbb{Q}$ is the additive identity.

   Given $\frac{m}{n} \in \mathbb{Q}$, the element $-\frac{m}{n} \in \mathbb{Q}$ is the additive inverse.

   The operation $+$ is associative (We know this because The operation $+$ is associative for ALL real numbers.)

4. The set $\mathbb{Q}'$ of all irrational numbers with operation addition.

   $(\mathbb{Q}', +)$ is **NOT a group.**

   The operation $+$ is **not closed** on $\mathbb{Q}'$. (To see this, observe that the sum of irrational numbers $0.10110111011110\ldots$ and $0.01001000100001\ldots$ is the rational number $0.11111111111111\ldots$)

   Since the *rational* number 0 is the additive identity, $(\mathbb{Q}', +)$ has **no additive identity.**

5. The set of all positive irrational numbers with operation multiplication.

   $\left((\mathbb{Q}')^+, \cdot\right)$ is **NOT a group.**

   The operation $\cdot$ is **not closed** on $\mathbb{Q}'$. (To see this, observe that the product of irrational numbers $\sqrt{2}$ and $\sqrt{2}$ is the rational number 2)

   Since the *rational* number 1 is the multiplicative identity, $\left((\mathbb{Q}')^+, \cdot\right)$ has **no additive identity.**

6. The set $\mathbb{Q}^+$ of all positive rational numbers with operation multiplication.

   $(\mathbb{Q}^+, \cdot)$ **IS a group.**

   The operation $\cdot$ is closed on $\mathbb{Q}^+$, since the product of two positive rational numbers is also a positive rational number.

   $1 \in \mathbb{Q}^+$ is the multiplicative identity.

   Given $\frac{m}{n} \in \mathbb{Q}^+$, the element $\frac{n}{m} \in \mathbb{Q}^+$ is the multiplicative inverse.

   The operation $\cdot$ is associative (We know this because The operation $\cdot$ is associative for ALL real numbers.)

7. The set of all real numbers x such that $0 < x \leq 1$ with operation multiplication.

   $(\{x \in \mathbb{R} : 0 < x \leq 1\}, \cdot)$ **is NOT a group.**

   The operation $\cdot$ IS closed on $\{x \in \mathbb{R} : 0 < x \leq 1\}$, since the product of two elements of $\{x \in \mathbb{R} : 0 < x \leq 1\}$ is again an element of $\{x \in \mathbb{R} : 0 < x \leq 1\}$.

   $1 \in \{x \in \mathbb{R} : 0 < x \leq 1\}$ is the multiplicative identity.

   The operation $\cdot$ is associative (We know this because The operation $\cdot$ is associative for ALL real numbers.)

   Given $x \in \{x \in \mathbb{R} : 0 < x \leq 1\}$, the element $\frac{1}{x}$ is the multiplicative inverse. HOWEVER, $\frac{1}{x} \notin \{x \in \mathbb{R} : 0 < x \leq 1\}$.

8. The set $\mathbf{E}$ of all even integers with operation addition.

   $(\mathbf{E}, +)$ **IS a group.**

   The operation $+$ is closed on $\mathbf{E}$, since the sum of two even numbers is also an even number.

   $0 \in \mathbf{E}$ is the additive identity

   Given the even number $2n \in \mathbf{E}$, the even number $-2n \in \mathbf{E}$ is the additive inverse.

   The operation $+$ is associative (We know this because The operation $+$ is associative for ALL real numbers.)

9. The set $\mathbf{E}$ of all even integers with operation multiplication.

   $(\mathbf{E}, \cdot)$ **is NOT a group.**

   The operation $\cdot$ IS closed on $\mathbf{E}$, since the product of two even numbers is also an even number.

   HOWEVER, the multiplicative identity $1 \notin \mathbf{E}$.

   ALSO, given the element $2n \in \mathbf{E}$, the multiplicative inverse, $\frac{1}{2n} \notin \mathbf{E}$.

10. The set of all multiples of a positive integer n with operation addition.

    The set is denoted $n\mathbb{Z} = \{0, \pm n, \pm 2n, \pm 3n, \ldots\}$

    $(n\mathbb{Z}, +)$ **IS a group.**

    $+$ is closed on $n\mathbb{Z}$. To see this, observe that given two elements jn and kn in $n\mathbb{Z}$, their sum $jn + kn = (j + k)\, n \in n\mathbb{Z}$

    The element $0 = 0n \in n\mathbb{Z}$ is the additive identity.

    Given $kn \in n\mathbb{Z}$, the element $-kn \in n\mathbb{Z}$ is the additive inverse.

    The operation $+$ is associative (We know this because The operation $+$ is associative for ALL real numbers.)

11. The set of all multiples of a positive integer n with operation multiplication.

    $(n\mathbb{Z}, \cdot)$ **is NOT a group.**

    $\cdot$ is closed on $n\mathbb{Z}$. To see this, observe that given two elements jn and kn in $n\mathbb{Z}$, their product $(jn)\,(kn) = (jkn)\, n \in n\mathbb{Z}$

    HOWEVER, the multiplicative identity $1 \notin n\mathbb{Z}$ for $n \neq 1$.

    ALSO, given $n \in n\mathbb{Z}$, the multiplicative inverse $\frac{1}{n} \notin n\mathbb{Z}$ for $n \neq 1$.

In Exercises 12-13, the given table defines an operation of multiplication on the set $S = \{e, a, b, c\}$. In each case, find a group axiom that fails to hold, and thereby show that S is not a group.

12.

$\times$	e	a	b	c
e	e	a	b	c
a	a	b	a	b
b	b	c	b	c
c	c	e	c	e

Here are a few things:

Notice that the identity element e does not appear in the row headed by a. This means that a does not have a right inverse.

Notice that the identity element e does not appear in the row or column headed by b. This means that b has neither a right inverse nor a left inverse.

Notice that the identity e appears twice in the row headed by c – once in the column headed by a and once in the column headed by c. This means that both a and c are right inverses of a, violating the fact that an inverse is unique.

13.

$\times$	e	a	b	c
e	e	a	b	c
a	e	a	b	c
b	e	a	b	c
c	e	a	b	c

Here are a few things:

All entries in the column headed by e show that e is NOT a right identity. (i.e., $xe \neq x$ for any element except $x = e$.) So, **there is NO two-sided identity.**

The fact that $xy = y$ for all elements $x, y \in S$, tell us that each element $x \in S$ is a left identity, contradicting the fact that such an identity should be unique.

The facts that:

$xa \neq e, \forall x \in S$

$xb \neq e, \forall x \in S$

$xc \neq e, \forall x \in S$

tell us that none of the elements a, b, c have a left inverse.