MTH 4441 HW Groups and Abelian Groups - Solutions

Fall 2017

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1. Part of the multiplication group table for the group $G = \{a, b, c, d\}$ is given. Complete the table.

$\times$	a	b	c	d
a		d		
b				
c			c	
d				c

First, observe that each element of the group must appear exactly once in each row and in each column.

Next, observe that c must be the group identity.

We arrive at this conclusion because $c \times c = c$. (eq. 1)

Since each element of the group must have a multiplicative inverse, c must have an inverse. We'll call it c^{-1}.

Multiplying each side of eq. 1 by c^{-1}, we have:

$c^{-1} \times (c \times c) = c^{-1} \times c \Rightarrow \left(c^{-1} \times c\right) \times c = c^{-1} \times c \Rightarrow e \times c = e \Rightarrow c = e$

Recognizing that c is the identity, the table becomes:

$\times$	a	b	c	d
a		d	a	
b			b	
c	a	b	c	d
d			d	c

Finally, observing the restriction that each element of a group table must appear exactly once in each row and column, the table becomes:

$\times$	a	b	c	d
a	c	d	a	b
b	d	c	b	a
c	a	b	c	d
d	b	a	d	c

2. Part of the multiplication group table for the group $G = \{a, b, c, d\}$ is given. Complete the table.

$\times$	a	b	c	d
a				
b		a		
c	a			
d				

Observe that c must be the group identity.

We arrive at this conclusion because $c \times a = a$. (eq. 2)

Since each element of the group must have a multiplicative inverse, a must have an inverse. We'll call it a^{-1}.

Multiplying each side of eq. 1 by a^{-1}, we have:

$(c \times a) \times a^{-1} = a \times a^{-1} \Rightarrow c \times \left(a \times a^{-1}\right) = a \times a^{-1} \Rightarrow c \times e = e \Rightarrow c = e$

Recognizing that c is the identity, the table becomes:

$\times$	a	b	c	d
a			a	
b		a	b	
c	a	b	c	d
d			d	

Finally, observing the restriction that each element of a group table must appear exactly once in each row and column, the table becomes:

$\times$	a	b	c	d
a	c	d	a	b
b	d	a	b	c
c	a	b	c	d
d	b	c	d	a

3. Prove that if $x = x^{-1}$ for all x in the group G, then G is abelian. (i.e., the binary operator is commutative.)

pf/
Suppose that $x = x^{-1}$ for all x in a group G.

If the group G contains exactly one element e, the G is abelian trivially.

So suppose the G has at least two *distinct* elements a and b, and that for any two elements a, b in a group, $(ab)^{-1} = \left(b^{-1}a^{-1}\right)$.

Observe: $\underbrace{ab = (ab)^{-1}}_{\text{because } x = x^{-1} \ \forall x \in G} = \left(b^{-1}a^{-1}\right) = ba$

i.e., $ab = ba$ ■

4. Suppose that $ab = ca$ implies that $b = c$ for all elements a, b, and c in a group G. Prove that G is abelian.

   pf/

   Suppose that $ab = ca$ implies that $b = c$ for all elements a, b, and c in a group G.

   Consider the product ab, for some $a, b \in G$

   Claim: For some element $c \in G$, $ab = ca$ (eq. 1)

   The easiest way to see this to recall that every group element must appear in every group table row and every group table column exactly once. In particular, the element ab must appear somewhere in the "a column." (i.e. ab must equal ca for some element $c \in G$.)

   By our hypothesis, this implies that $b = c$.

   Thus we can substitute b for c in eq. 1, yielding $ab = ba$.

   i.e., $ab = ba \ \forall a, b \in G$.

   Hence, G is abelian. ■

5. Let a, b be elements of a group G. Prove that G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$.

pf/

$\boxed{G \text{ is abelian} \Rightarrow (ab)^{-1} = a^{-1}b^{-1},\ \ \forall a, b \in G.}$

Suppose that G is abelian. Then the binary operator is commutative. (i.e., $xy = yx$ for all $x, y \in G$.)

Hence, $(ab)^{-1} = \underbrace{b^{-1}a^{-1} = a^{-1}b^{-1}}_{\text{by commutativity}}$

i.e., $(ab)^{-1} = a^{-1}b^{-1}$

$\boxed{(ab)^{-1} = a^{-1}b^{-1},\ \ \forall a, b \in G \Rightarrow G \text{ is abelian.}}$

Let the hypothesis be given. (i.e., Suppose that $(ab)^{-1} = a^{-1}b^{-1},\ \ \forall a, b \in G$.)

Let $a, b \in G$.

Observation #1:

For **any** group G and for any $a, b \in G$, $(ab)^{-1} = b^{-1}a^{-1}$ – **always!**

Also:

$\left(a^{-1}\right)^{-1} = a$ and $\left(b^{-1}\right)^{-1} = b\ \ \forall a, b \in G$.

Finally Observe:

$$\underbrace{\left(a^{-1}b^{-1}\right)^{-1} = \left(a^{-1}\right)^{-1}\left(b^{-1}\right)^{-1}}_{\text{By our hypothesis}} = ab$$

$$\underbrace{\left(a^{-1}b^{-1}\right)^{-1} = \left(b^{-1}\right)^{-1}\left(a^{-1}\right)^{-1}}_{\text{By Observation \#1}} = ba$$

$\Rightarrow ab = \left(a^{-1}b^{-1}\right)^{-1} = ba$

i.e., $ab = ba$

Hence, G is abelian.

6. Let a, b be elements of a group G. Prove that G is abelian if and only if $(ab)^2 = a^2b^2$.

pf/

$\boxed{G \text{ is abelian} \Rightarrow (ab)^2 = a^2b^2, \ \ \forall a, b \in G.}$

Suppose that G is abelian. Then the binary operator is commutative.

Observe:

$$(ab)^2 = (ab)(ab) = \underbrace{a(ba)b = a(ab)b}_{\text{by commutativity, } ba = ab} = (aa)(bb) = a^2b^2$$

i.e., $(ab)^2 = a^2b^2$

$\boxed{(ab)^2 = a^2b^2, \ \ \forall a, b \in G \Rightarrow G \text{ is abelian.}}$

Suppose that $(ab)^2 = a^2b^2, \ \ \forall a, b \in G$

Observe:

$(ab)(ab) = (ab)^2 = a^2b^2$

i.e., $abab = a^2b^2$

$\Rightarrow a^{-1}(abab) = a^{-1}(a^2b^2)$

$\Rightarrow (a^{-1}a)\,bab = (a^{-1}a^2)\,b^2$
$\Rightarrow bab = ab^2$

$\Rightarrow (bab)\,b^{-1} = (ab^2)\,b^{-1}$

$\Rightarrow ba\,(bb^{-1}) = a\,(b^2b^{-1})$

$\Rightarrow ba = ab$

i.e., $ab = ba$

Hence G is abelian. $\blacksquare$