

MTH 4441 Homework #2 Groups - Solutions

FALL 2017

Pat Rossi

Name _____

For Exercises 1-10, decide whether each of the given sets is a group with respect to the given operation. If it is NOT a group, state at least one of the group axioms that fails to hold.

Group Axioms for $(G, *)$

- The Binary Operator $*$ is **closed** on G .
- $*$ is associative
- $(G, *)$ has an identity element
- Each element $x \in G$ has an inverse.

1. The set \mathbb{Z}^+ of all positive integers with operation addition.

$(\mathbb{Z}^+, +)$ is **NOT a group**. $(\mathbb{Z}^+, +)$ has no additive identity. (The additive identity would be 0, but $0 \notin \mathbb{Z}^+$.)

Also: The additive inverse of each element $n \in \mathbb{Z}^+$ is the *negative* integer $-n$, which is NOT an element of \mathbb{Z}^+ .

2. The set \mathbb{Z}^+ of all positive integers with operation multiplication.

(\mathbb{Z}^+, \cdot) is **NOT a group**. The multiplicative inverse of each element $n \in \mathbb{Z}^+$ is the rational number $\frac{1}{n}$, which is NOT an element of \mathbb{Z}^+ for $n > 1$.

3. The set \mathbb{Q} of all rational numbers with operation addition.

$(\mathbb{Q}, +)$ **IS a group**.

The operation $+$ is closed on \mathbb{Q} , since the sum of two rational numbers is also a rational number.

$0 \in \mathbb{Q}$ is the additive identity.

Given $\frac{m}{n} \in \mathbb{Q}$, the element $-\frac{m}{n} \in \mathbb{Q}$ is the additive inverse.

The operation $+$ is associative (We know this because The operation $+$ is associative for ALL real numbers.)

4. The set \mathbb{Q}' of all irrational numbers with operation addition.

$(\mathbb{Q}', +)$ is **NOT a group**.

The operation $+$ is **not closed** on \mathbb{Q}' . (To see this, observe that the sum of irrational numbers $0.10110111011110\dots$ and $0.01001000100001\dots$ is the rational number $0.11111111111111\dots$)

Also: Since the *rational* number 0 is the additive identity for ALL subsets of the Real Numbers under addition, $(\mathbb{Q}', +)$ has **no additive identity**, since $0 \notin \mathbb{Q}'$

5. The set of all positive irrational numbers with operation multiplication.

$((\mathbb{Q}')^+, \cdot)$ is **NOT a group**.

The operation \cdot is **not closed** on \mathbb{Q}' . (To see this, observe that the product of irrational numbers $\sqrt{2}$ and $\sqrt{2}$ is the rational number 2)

Also: Since the *rational* number 1 is the multiplicative identity for ALL subsets of the Real Numbers under multiplication, $((\mathbb{Q}')^+, \cdot)$ has **no additive identity**, since $1 \notin \mathbb{Q}'$.

6. The set \mathbb{Q}^+ of all positive rational numbers with operation multiplication.

(\mathbb{Q}^+, \cdot) **IS a group**.

The operation \cdot is closed on \mathbb{Q}^+ , since the product of two positive rational numbers is also a positive rational number.

$1 \in \mathbb{Q}^+$ is the multiplicative identity.

Given $\frac{m}{n} \in \mathbb{Q}^+$, the element $\frac{n}{m} \in \mathbb{Q}^+$ is the multiplicative inverse.

The operation \cdot is associative (We know this because The operation \cdot is associative for ALL real numbers.)

7. The set \mathbf{E} of all even integers with operation addition.

$(\mathbf{E}, +)$ **IS a group**.

The operation $+$ is closed on \mathbf{E} , since the sum of two even numbers is also an even number.

$0 \in \mathbf{E}$ is the additive identity

Given the even number $2n \in \mathbf{E}$, the even number $-2n \in \mathbf{E}$ is the additive inverse.

The operation $+$ is associative (We know this because The operation $+$ is associative for ALL real numbers.)

8. The set \mathbf{E} of all even integers with operation multiplication.

(\mathbf{E}, \cdot) **is NOT a group**.

The operation \cdot IS closed on \mathbf{E} , since the product of two even numbers is also an even number.

HOWEVER, the multiplicative identity $1 \notin \mathbf{E}$.

ALSO, given the element $2n \in \mathbf{E}$, the multiplicative inverse, $\frac{1}{2n} \notin \mathbf{E}$.

9. The set of all multiples of 5 with operation addition.

The set is denoted $5\mathbb{Z} = \{0, \pm 5, \pm 10, \pm 15, \dots\}$

$(5\mathbb{Z}, +)$ **IS a group.**

$+$ is closed on $5\mathbb{Z}$. To see this, observe that given two elements $5j$ and $5k$ in $5\mathbb{Z}$, their sum $5j + 5k = 5(j + k) \in 5\mathbb{Z}$

The element $0 = 5 \cdot 0 \in 5\mathbb{Z}$ is the additive identity.

Given $5k \in 5\mathbb{Z}$, the element $-5k \in 5\mathbb{Z}$ is the additive inverse.

The operation $+$ is associative (We know this because The operation $+$ is associative for ALL real numbers.)

10. The set of all multiples of 5 with operation multiplication.

$(5\mathbb{Z}, \cdot)$ **is NOT a group.**

\cdot is closed on $5\mathbb{Z}$. To see this, observe that given two elements $5j$ and $5k$ in $5\mathbb{Z}$, their product $(5j)(5k) = 25jk = 5(5jk) \in 5\mathbb{Z}$

HOWEVER, the multiplicative identity $1 \notin 5\mathbb{Z}$.

ALSO, given $n \in 5\mathbb{Z}$, the multiplicative inverse $\frac{1}{n} \notin 5\mathbb{Z}$.

In Exercises 11-12, the given table defines an operation of multiplication on the set $S = \{e, a, b, c\}$. In each case, find a group axiom that fails to hold, and thereby show that S is **not** a group.

11.

\cdot	e	a	b	c
e	e	a	b	c
a	a	b	a	b
b	b	c	b	c
c	c	e	c	e

Here are a few things:

Notice that the identity element e does not appear in the row headed by a . This means that a does not have a right inverse.

Notice that the identity element e does not appear in the row or column headed by b . This means that b has neither a right inverse nor a left inverse.

Notice that the identity e appears twice in the row headed by c – once in the column headed by a and once in the column headed by c . This means that both a and c are right inverses of c , violating the fact that an inverse is unique.

12.

\cdot	e	a	b	c
e	e	a	b	c
a	e	a	b	c
b	e	a	b	c
c	e	a	b	c

Here are a few things:

All entries in the column headed by e show that e is NOT a right identity. (i.e., $xe \neq x$ for any element except $x = e$.) So, **there is NO two-sided identity.**

The fact that $xy = y$ for all elements $x, y \in S$, tell us that each element $x \in S$ is a left identity, contradicting the fact that such an identity should be unique.

The facts that:

$$xa \neq e, \forall x \in S$$

$$xb \neq e, \forall x \in S$$

$$xc \neq e, \forall x \in S$$

tell us that none of the elements a, b, c have a left inverse.

In exercises, 13-18, let the binary operation be defined on \mathbb{Z} by the rule given. Determine in each case whether $(\mathbb{Z}, *)$ is a group. If it is a group, determine if it is an abelian group. State which conditions, if any fail to hold.

13. $x * y = x + y + 1$

This IS a group.

$*$ is closed on \mathbb{Z}

Find the identity: We want $e \in \mathbb{Z}$ such that $x * e = x + e + 1 = x$

$$\Rightarrow x + e + 1 = x \Rightarrow e + 1 = 0 \Rightarrow e = -1$$

Check: $x * e = x + e + 1 = x + (-1) + 1 = x$

Also: $e * x = e + x + 1 = (-1) + x + 1 = x$

i.e. For $e = -1$, we have: $e * x = x = x * e$ (i.e. $e = -1$ IS the identity)

Find the inverse: (i.e., For $x \in \mathbb{Z}$, find x^{-1})

We want x^{-1} such that $x * x^{-1} = x + x^{-1} + 1 = e = -1$

$$\Rightarrow x + x^{-1} + 1 = -1 \Rightarrow x + x^{-1} = -2 \Rightarrow x^{-1} = -2 - x$$

Check: $x * x^{-1} = x + x^{-1} + 1 = x + (-2 - x) + 1 = -1 = e$

Also: $x^{-1} * x = x^{-1} + x + 1 = (-2 - x) + x + 1 = -1 = e$

i.e., $x * x^{-1} = e = x^{-1} * x$ (i.e., Given $x \in \mathbb{Z}$, $x^{-1} = -2 - x$)

Regarding **associativity:**

$$\begin{aligned} (x * y) * z &= (x + y + 1) * z = (x + y + 1) + z + 1 = x + y + z + 1 + 1 = x + (y + z + 1) + 1 \\ &= x * (y + z + 1) = x * (y * z) \end{aligned}$$

i.e., $(x * y) * z = x * (y * z)$ ($*$ is associative)

Furthermore, $(\mathbb{Z}, *)$ **IS an abelian group.**

This will follow, if we can show that $x * y = y * x$

Observe: $x * y = (x + y + 1) = (y + x + 1) = y * x$

14. $x * y = x + y - 1$

This IS a group.

$*$ is closed on \mathbb{Z}

$e = 1$ is the identity

Find the identity: We want $e \in \mathbb{Z}$ such that $x * e = x + e - 1 = x$

$$\Rightarrow x + e - 1 = x \Rightarrow e - 1 = 0 \Rightarrow e = 1$$

Check: $x * e = x + e - 1 = x + (1) - 1 = x$

Also: $e * x = e + x - 1 = (1) + x - 1 = x$

i.e. For $e = 1$, we have: $e * x = x = x * e$ (i.e. $e = 1$ IS the identity)

Find the inverse: (i.e., For $x \in \mathbb{Z}$, find x^{-1})

We want x^{-1} such that $x * x^{-1} = x + x^{-1} - 1 = e = 1$

$$\Rightarrow x + x^{-1} - 1 = 1 \Rightarrow x + x^{-1} = 2 \Rightarrow x^{-1} = 2 - x$$

Check: $x * x^{-1} = x + x^{-1} - 1 = x + (2 - x) - 1 = 1 = e$

Also: $x^{-1} * x = x^{-1} + x - 1 = (2 - x) + x - 1 = 1 = e$

i.e., $x * x^{-1} = e = x^{-1} * x$ (i.e., Given $x \in \mathbb{Z}$, $x^{-1} = 2 - x$)

Regarding **Associativity:**

$$\begin{aligned} (x * y) * z &= (x + y - 1) * z = (x + y - 1) + z - 1 = x + y + z - 1 - 1 = x + (y + z - 1) - 1 \\ &= x * (y + z - 1) = x * (y * z) \end{aligned}$$

i.e., $(x * y) * z = x * (y * z)$ ($*$ is associative)

Furthermore, $(\mathbb{Z}, *)$ **IS an abelian group.**

This will follow, if we can show that $x * y = y * x$

Observe: $x * y = (x + y - 1) = (y + x - 1) = y * x$

15. $x * y = x + xy$

This is NOT a group.

$*$ is closed on \mathbb{Z}

Is there an identity? (i.e., is there an element e such that $e * x = x = x * e$?)

Observe: $x * e = x + xe = x, \Rightarrow xe = 0$

$$\Rightarrow xe = 0 \Rightarrow e = 0$$

i.e., $x * 0 = x$, so $e = 0$ may be the identity.

Let's check to see if $0 * x = x$.

$$0 * x = 0 + 0 \cdot x = 0.$$

i.e., $x * 0 = x$, but $0 * x = 0$.

So $e = 0$ is NOT a (two-sided) identity

i.e., **There is no identity**

Hence, given $x \in \mathbb{Z}$, there can be no x^{-1} .

Given $x \in \mathbb{Z}$, x^{-1} **Does Not Exist**)

Regarding Associativity:

$$(x * y) * z = (x + xy) * z = (x + xy) + (x + xy)z = x + xy + xz + xyz$$

$$x * (y * z) = x * (y + yz) = x + x(y + yz) = x + xy + xyz$$

i.e., $(x * y) * z \neq x * (y * z)$ (*** is NOT associative**)

16. $x * y = xy + y$

This is NOT a group.

* is closed on \mathbb{Z}

Is there an identity? (i.e., is there an element e such that $e * x = x = x * e$?)

Observe: $x * e = xe + e = x \Rightarrow (x + 1)e = x \Rightarrow e = \frac{x}{x+1}$

i.e., $e = \frac{x}{x+1}$

Note that e is NOT a constant value - it's value depends on the value of x . So there is no single element e such that $x * e = x$.

Therefore, **there is no identity**

Hence, given $x \in \mathbb{Z}$, there can be no x^{-1} .

Given $x \in \mathbb{Z}$, x^{-1} **Does Not Exist**)

Regarding Associativity:

$$(x * y) * z = (xy + y) * z = (xy + y)z + z = xyz + yz + z$$

$$x * (y * z) = x * (yz + z) = x(yz + z) + (yz + z) = xyz + xz + yz + z$$

i.e., $(x * y) * z \neq x * (y * z)$ (*** is NOT associative**)

17. $x * y = x + xy + y$

This is NOT a group.

* is closed on \mathbb{Z}

Is there an identity? (i.e., is there an element e such that $e * x = x = x * e$?)

Observe: $x * e = x + xe + e = x \Rightarrow xe + e = 0 \Rightarrow (x + 1)e = 0 \Rightarrow e = \frac{0}{x+1}$

i.e., $e = 0$

Also:

$e * x = e + ex + x = x \Rightarrow e + ex = 0 \Rightarrow e(1 + x) = 0 \Rightarrow e = \frac{0}{1+x}$

i.e., $e = 0$

Next, $\forall x \in \mathbb{Z}$, does there exist an x^{-1} ?

Consider:

$x * x^{-1} = x + xx^{-1} + x^{-1} = 0 \Rightarrow xx^{-1} + x^{-1} = -x \Rightarrow (x + 1)x^{-1} = -x$

$\Rightarrow x^{-1} = \frac{-x}{x+1}$

Note that a consequence of this result is that x^{-1} is undefined for $x = -1$. (i.e., $x = -1$ has no inverse.)

x inverse does not exist for every element in \mathbb{Z}

Regarding Associativity:

$$\begin{aligned} (x * y) * z &= (x + xy + y) * z = (x + xy + y) + (x + xy + y)z + z = x + xy + y + xz + xyz + yz + z \\ &= x + y + z + xy + xz + yz + xyz \end{aligned}$$

$$\begin{aligned} x * (y * z) &= x * (y + yz + z) = x + x(y + yz + z) + (y + yz + z) = x + xy + xyz + xz + y + yz + z \\ &= x + y + z + xy + xz + yz + xyz \end{aligned}$$

i.e., $(x * y) * z = x * (y * z)$ (*** IS associative**)

18. $x * y = x - y$

This is NOT a group.

* is closed on \mathbb{Z}

Is there an identity? (i.e., is there an element e such that $e * x = x = x * e$?)

Observe: $x * e = x - e = x \Rightarrow -e = 0 \Rightarrow e = 0$

i.e., $e = 0$

Also:

$e * x = e - x = x \Rightarrow e = 2x$

i.e., $e = 2x$

The left and right sided identities are not equal, so there is **NO Identity**.

Consequently, $\forall x \in \mathbb{Z}$, **there does NOT exist an inverse**.

Regarding Associativity:

$$(x * y) * z = (x + xy + y) * z = (x + xy + y) + (x + xy + y)z + z = xz + xyz + yz + z$$

$$x * (y * z) = x * (y + yz + z) = x + x(y + yz + z) + (y + yz + z) = x + xy + xyz + y + yz + z$$

i.e., $(x * y) * z \neq x * (y * z)$ (*** is NOT associative**)

In exercises, 19-21, Fill in the group table for $(G, *)$ in as many different ways as possible.

19.

*	e	a
e		
a		

*	e	a
e	e	a
a	a	e

This is the only possibility.

20.

*	e	a	b
e			
a			
b			

*	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

This is the only possibility.

21.

*	e	a	b	c
e				
a				
b				
c				

*	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

*	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	a	c
c	c	b	c	e

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

are all possibilities.