MTH 4441 HW #3 - SUBGROUPS - Solutions

Fall 2017

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1. Given the group table for $(G, *)$, find all of the subgroups of $(G, *)$ and justify your answers. Draw a subgroup diagram for $(G, *)$.

$*$	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

To start off, we acknowledge that $(\{e\}, *)$ and $(G, *)$ are subgroups of $(G, *)$.

If there are other subgroups $(H, *)$, then $|H|$ must divide $|G|$.

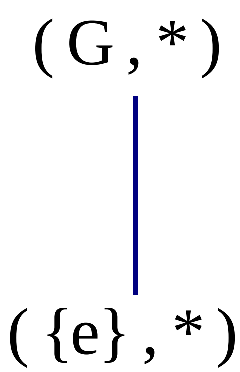
But $|G| = 5$

$\Rightarrow |H| = 1$ or $\Rightarrow |H| = 5$.

Thus, we have already accounted for all possible subgroups of $(G, *)$.

Hence $(\{e\}, *)$ and $(G, *)$ are the **only** subgroups of $(G, *)$.

Our subgroup diagram is below:



2. Given the group table for $(G, *)$, find all of the subgroups of $(G, *)$ and justify your answers. Draw a subgroup diagram for $(G, *)$.

$*$	e	v	w	x	y	z
e	e	v	w	x	y	z
v	v	w	x	y	z	e
w	w	x	y	z	e	v
x	x	y	z	e	v	w
y	y	z	e	v	w	x
z	z	e	v	w	x	y

To start off, we acknowledge that $(\{e\}, *)$ and $(G, *)$ are subgroups of $(G, *)$.

If there are other subgroups $(H, *)$, then $|H|$ must divide $|G|$.

Since $|G| = 6$, this implies that $|H| = 1, 2, 3$, or 6.

So we are looking for subgroups of order 2 or order 3.

Let's see what happens when we take $(H, *) = (\{e\}, *)$ and add a single element of G to H.

$v \in H$

If $(H, *)$ is a **subgroup** of $(G, *)$ and $v \in H$, then $(H, *)$ must also contain v^{-1}.

From the group table of $(G, *)$, $v^{-1} = z$.

Thus, $z \in H$.

Also, since $*$ must be closed on $(H, *)$, $v * v \in H$ and $v * (v * v) \in H$, etc.

Observe: $v * v = w$, and $v * (v * v) = v * w = x$, and $v * (v * v * v) = v * x = y$, and $v * (v * v * v * v) = v * z = z$

i.e., if $v \in H$, then $(H, *) = (\{e, v, w, x, y, z\}, *) = (G, *)$

$z \in H$

If $(H, *)$ is a **subgroup** of $(G, *)$ and $z \in H$, then $(H, *) = (G, *)$, for reasons similar to thise given in the preceding case.

$w \in H$

If $(H, *)$ is a **subgroup** of $(G, *)$ and $w \in H$, then $(H, *)$ must also contain w^{-1}.

From the group table of $(G, *)$, $w^{-1} = y$.

Thus, $y \in H$.

Also, since $*$ must be closed on $(H, *)$, $w * w \in H$ and $w * (w * w) \in H$, etc.

Observe: $w * w = y$, and $w * (w * w) = w * y = e$

Thus, $w, y \in (H, *)$.

A quick check shows that $*$ is closed on $\{e, w, y\}$, as $w * y = e$ and $y * w = e$ and $y * y = w$

Therefore, $(H, *) = (\{e, w, y\}, *)$ is a subgroup of $(G, *)$ having order 3.

$y \in H$

Reasoning similar to that used in the previous case yields the subgroup $(H, *) = (\{e, w, y\}, *)$ of order 3.

$x \in H$

If $(H, *)$ is a **subgroup** of $(G, *)$ and $x \in H$, then $(H, *)$ must also contain x^{-1}.

From the group table of $(G, *)$, $x^{-1} = x$.

Thus, $x \in H$.

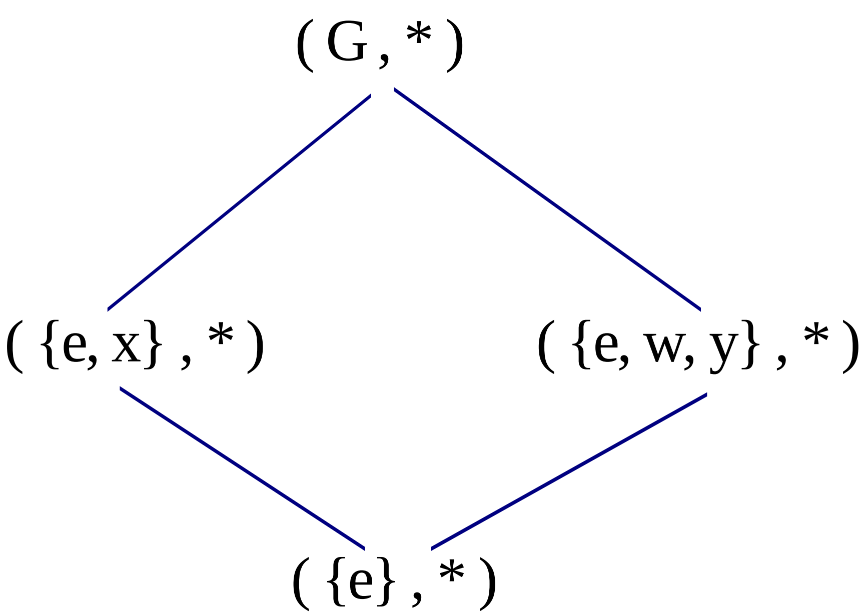
Also, since $x^{-1} = x$ $*$ is closed on $(H, *)$.

Therefore, $(H, *) = (\{e, x\}, *)$ is a subgroup of $(G, *)$ having order 2.

Thus, we have already accounted for all possible subgroups of $(G, *)$.

Hence $(\{e\}, *), (\{e, x\}, *), (\{e, w, y\}, *)$, and $(G, *)$ are the subgroups of $(G, *)$.

Our subgroup diagram is shown below:



3. Given the group table for $(G, *)$, find all of the subgroups of $(G, *)$ and justify your answers. Draw a subgroup diagram for $(G, *)$.

$*$	e	v	w	x	y	z
e	e	v	w	x	y	z
v	v	e	x	z	w	y
w	w	x	e	y	z	v
x	x	z	y	e	v	w
y	y	w	z	v	e	x
z	z	y	v	w	x	e

To start off, we acknowledge that $(\{e\}, *)$ and $(G, *)$ are subgroups of $(G, *)$.

If there are other subgroups $(H, *)$, then $|H|$ must divide $|G|$.

Since $|G| = 6$, this implies that $|H| = 1, 2, 3$, or 6.

So we are looking for subgroups of order 2 or order 3.

Let's see what happens when we take $(H, *) = (\{e\}, *)$ and add a single element of G to H.

Note that every element of $(G, *)$ is its own inverse.

Since each element in G is its own inverse:

(a) each of the prospective subgroups $(\{e, v\}, *)$; $(\{e, w\}, *)$; $(\{e, x\}, *)$; $(\{e, y\}, *)$; $(\{e, z\}, *)$ is such that each element has an inverse that is contained in the prospective subroup

(b) $*$ is closed on each of the prospective subgroups

Thus, $(\{e, v\}, *)$; $(\{e, w\}, *)$; $(\{e, x\}, *)$; $(\{e, y\}, *)$; $(\{e, z\}, *)$ are all subgroups (of order 2) of $(G, *)$.

Q: Can we create subgroups of order 3, by adding a single element to one of the subgroups of order 2?

In this particular case, this is the only possible way to create a subgroup of order 3.

Here's why:

Suppose that we DO have a subgroup of order 3. (For example, $(\{e, v, w\}, *)$.)

If we remove one of the "non-identity elements" from the subgroup (let's say that we remove w), what do we have left?

One of the subgroups of order 2 – $(\{e, v\}, *)$.

And how do we get back the "original" subgroup $(\{e, v, w\}, *)$ of order 3?

Add the element w to the subgroup $(\{e, v\}, *)$ of order 2.

Thus, if there ARE any subgroups of order 3, they can be formed by taking one of the subgroups of order 2 and adding an element to the subgroup.

Let's Try It!

What happens when we add, for example, the element w to the subgroup $(\{e, v\}, *)$?

Since $w = w^{-1}$, the prospective subgroup contains the inverses of all of its elements.

Is $(\{e, v, w\}, *)$ a subgroup of $(G, *)$?

No!

Here's why: $*$ is not closed on $(\{e, v, w\}, *)$.

Observe that $v * w = x$ and $w * v = x$.

But $x \notin (\{e, v, w\}, *)$.

Remark: We could try all of the other possibilities, one by one, and we will end up with the same result – $*$ will not be closed on the prospective subgroup. Instead of going through the possibilities one by one, let's see why adding **any** element to an existing subgroup of order 2 will fail to produce a subgroup of order 3. In each case, closure of $*$ on the prospective subgroup will be the issue.

Without loss of generality, consider again the subgroup $(\{e, v\}, *)$ of order 2. And again let's add the element w, forming the prospective subgroup $(\{e, v, w\}, *)$ of order 3.

Without actually computing the product $v*w$ or the product $w*v$, let's see why it would be impossible for $*$ to be closed on $\{e, v, w\}$.

In order for $*$ to be closed on $\{e, v, w\}$, $v * w$ must equal either e, v, or w.

But $v * w$ can't be equal to e, because w is not v^{-1}.

And $v * w$ can't be equal to v, because w is not the identity.

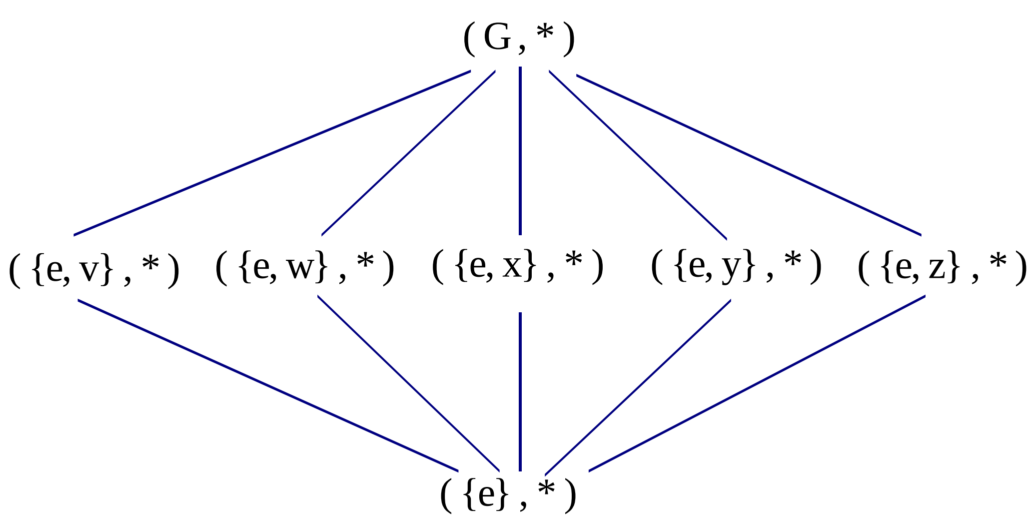
And $v * w$ can't be equal to w, because v is not the identity.

The same exact reasoning covers all possible cases of adding a third element to an existing subgroup of order 2, in an attempt to form a subgroup of order 3.

So there are NO subgroups of order 3.

The subgroups of $(G, *)$ are $(\{e\}, *)$; $(\{e, v\}, *)$; $(\{e, w\}, *)$; $(\{e, x\}, *)$; $(\{e, y\}, *)$; $(\{e, z\}, *)$; and $(G, *)$

Our subgroup diagram is shown below:



4. Recall that $(\mathbb{Z},+)$ is a group with identity 0, and that $(\{0\},+)$ and $(\mathbb{Z},+)$ must be subgroups of $(\mathbb{Z},+)$.

   Recall that $(2\mathbb{Z},+)$, where $2\mathbb{Z}=\{0,\pm2,\pm4,\pm6,\ldots,\pm2k,\ldots\}=\{2k:k\in\mathbb{Z}\}$, is also a subgroup of $(\mathbb{Z},+)$.

   Does $(\mathbb{Z},+)$ have any subgroups that are also subgroups of $(2\mathbb{Z},+)$?

   Observe: $(4\mathbb{Z},+)$ is a subgroup of $(2\mathbb{Z},+)$ because:

   i) $(4\mathbb{Z},+)$ is a group in its own right, and

   ii) any element of $4n\in4\mathbb{Z}$ can be written as $4n=2(2n)$, and hence, is an element of $2\mathbb{Z}$.

   Thus, $(4\mathbb{Z},+)\leq(2\mathbb{Z},+)$.

   Similarly, $(6\mathbb{Z},+);(8\mathbb{Z},+);(10\mathbb{Z},+);(12\mathbb{Z},+);\ldots$ are subgroups of $(2\mathbb{Z},+)$.

5. Recall that $(\mathbb{Q}^+,\cdot)$ is a group with identity 1, and that $(\{1\},\cdot)$ and $(\mathbb{Q}^+,\cdot)$ must be subgroups of $(\mathbb{Q}^+,\cdot)$.

   Recall that $\left(\left\{1,2^{\pm1},2^{\pm2},2^{\pm3},\ldots,2^{\pm k},\ldots\right\},\cdot\right)$, is also a subgroup of $(\mathbb{Q}^+,\cdot)$.

   Does $(\mathbb{Q}^+,\cdot)$ have any subgroups that are also subgroups of $\left(\left\{1,2^{\pm1},2^{\pm2},2^{\pm3},\ldots,2^{\pm k},\ldots\right\},\cdot\right)$?

   Observe: $\left(\left\{1,4^{\pm1},4^{\pm2},4^{\pm3},\ldots,4^{\pm k},\ldots\right\},\cdot\right)$ is a subgroup of $\left(\left\{1,2^{\pm1},2^{\pm2},2^{\pm3},\ldots,2^{\pm k},\ldots\right\},\cdot\right)$ because:

   i) $\left(\left\{1,4^{\pm1},4^{\pm2},4^{\pm3},\ldots,4^{\pm k},\ldots\right\},\cdot\right)$ is a group in its own right, and

   ii) any element of $4^{\pm n}\in\left(\left\{1,4^{\pm1},4^{\pm2},4^{\pm3},\ldots,4^{\pm k},\ldots\right\},\cdot\right)$ can be written as $4^{\pm n}=\left(2^2\right)^{\pm n}=2^{\pm2n}$, and hence, is an element of $\left(\left\{1,2^{\pm1},2^{\pm2},2^{\pm3},\ldots,2^{\pm k},\ldots\right\},\cdot\right)$.

   Thus, $\left(\left\{1,4^{\pm1},4^{\pm2},4^{\pm3},\ldots,4^{\pm k},\ldots\right\},\cdot\right)\leq\left(\left\{1,2^{\pm1},2^{\pm2},2^{\pm3},\ldots,2^{\pm k},\ldots\right\},\cdot\right)$.

   Similarly, $\left(\left\{1,8^{\pm1},8^{\pm2},8^{\pm3},\ldots\right\},\cdot\right);\left(\left\{1,16^{\pm1},16^{\pm2},16^{\pm3},\ldots\right\},\cdot\right);\left(\left\{1,32^{\pm1},32^{\pm2},32^{\pm3},\ldots\right\},\cdot\right);\ldots$ are subgroups of $\left(\left\{1,2^{\pm1},2^{\pm2},2^{\pm3},\ldots,2^{\pm k},\ldots\right\},\cdot\right)$.