MTH 4441 HW #5 - Isomorphisms - Solutions

Fall 2017

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1. Show that $(\mathbb{Z}_n, +) = (\{0, 1, 2, \ldots, n-1\}, +)$, where "+" is addition modulo n, is a group.

2. Construct the group table for $(\mathbb{Z}_6, +)$

$(\mathbb{Z}_6, +)$

+	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

3. In the preceding example:

   (a) What is the inverse of 2?

   (b) What is the inverse of 3?

4. Construct the group table for $(\mathbb{Z}_4, +)$

$(\mathbb{Z}_4, +)$

+	0	1	2	3
0				
1				
2				
3				

5. Construct the group table for $(U_5, \cdot)$

$(U_5, \cdot)$

$\cdot$	1	2	3	4
1				
2				
3				
4				

6. With reference to Exercises 4 and 5, show that the two groups are isomorphic by defining an isomorphism $\phi : (\mathbb{Z}_4, +) \to (U_5, \cdot)$

7. Show that the function ϕ in the previous exercise is an isomorphism.

8. As incredible as it seems, the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$, where "+" and "$\cdot$" are the usual addition and multiplication of real numbers, are isomorphic. Show that $\phi : (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$, given by $\phi(x) = e^x$, is an isomorphism. (the function e^x is the exponential function (that we all know and love) from Calculus.)

Relating to Exercises 9-11, Given sets S_1 and S_2, the **product** of S_1 and S_2, denoted $S_1 \times S_2$, is given by:

$$S_1 \times S_2 = \{(x, y) : x \in S_1 \text{ and } y \in S_2\}$$

Addition of elements in $S_1 \times S_2$ is done component-wise: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

9. Construct the group table for $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$, starting with:

10. Recall from class lectures that $(G_3, *_3)$ was not isomorphic to either $(G_1, *_1) = (U_5, \cdot)$ or $(G_2, *_2)$ (all shown below). Show that $(G_3, *_3)$ IS isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ (of the previous exercise) by exhibiting an isomorphism $\phi : (G_3, *_3) \to (\mathbb{Z}_2 \times \mathbb{Z}_2, +)$.

$(G_1, *_1) = (U_5, \cdot)$

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$(G_2, *_2)$

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

$(G_3, *_3)$

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

11. Show that $(G_3, *_3) \to (\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ (as defined in the previous exercise) IS an isomorphism.

12. From previous examples, list all algebraic properties that isomorphisms preserve and all properties that isomorphic groups have in common.