HW #6 - Cyclic Groups

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Pat Rossi Name ______________________

Remark: In doing these homework exercises, we may choose to make use of any of the theorems listed below:

Cyclic Groups In General

Thm 1 - every subgroup of a cyclic group is cyclic

Thm 2 - Any two finite cyclic groups of order n are isomorphic (i.e., Any two finite cyclic groups of the same order are isomorphic.)

Thm 3 - An isomorphism ϕ between two finite cyclic groups of order n is completely defined by the value of $\phi(a)$, where a is a generator (*any* generator) of the first group. Since isomorphisms map "generator to generator," $\phi(a)$ must be a generator (*any* generator) of the second group. The value of $\phi(b)$ for any element b of the first group is completely determined by the value of $\phi(a)$.

Thm 4 - The property of two groups being isomorphic is an equivalence relationship. That is to say:

i) $(G,*) \cong (G,*)$

ii) $(G,*_1) \cong (H,*_2) \Rightarrow (H,*_2) \cong (G,*_1)$

iii) If $(G,*_1) \cong (H,*_2)$ and $(H,*_2) \cong (K,*_3)$, then $(G,*_1) \cong (K,*_3)$

Cyclic Groups With "Additive Notation"

Thm 5 - The generators of the cyclic group $(\mathbb{Z}_n,+)$ are exactly those non-zero "proper remainders" $\{1,2,3,\ldots,n-1\}$ that are relatively prime to n.

Thm 6 - Given the cyclic group $(\mathbb{Z}_n,+)$, if $a \in \{1,2,3,\ldots,n-1\}$, then a generates a cyclic subgroup of order $\frac{|G|}{d}$ where $d = \gcd(a,n)$.

Cyclic Groups With "Multiplicative Notation"

Thm 7 - $k \in U_n$ is a generator of $(U_n,\cdot)$ exactly when n is the least positive integer such that $k^{n-1} \equiv 1 \bmod n$

Cor - $k \in U_n$ is a generator of $(U_n,\cdot)$ exactly when n is the least positive integer such that $k^{\frac{n-1}{2}} \equiv n-1 \bmod n$

Thm 8 - Let $(G,*) = \langle a \rangle$ be a finite cyclic group of order n. Then a^m is a generator of G exactly when m and n are relatively prime. (i.e., exactly when $\gcd(m,n)=1$).

Thm 9 - If G is cyclic with generator a, and $H < G$, then either:

a. $H = \langle e \rangle$ (i.e., $H = (\{e\},*)$

or

b. $H = \langle a^k \rangle$, where k is the least natural number such that $a^k \in H$.

Thm 10 - Let $G = \langle a \rangle$ be a *finite* cyclic group of order n. For any integer m, $\langle a^m \rangle = \langle a^d \rangle$, where $d = \gcd(m,n)$.

Thm 11 - Suppose that G is a finite cyclic group of order n. Then:

i. for any generator $a \in G$, n is the least natural number such that $a^n = e$.

and

ii. if $a^s = a^t$, then $s \equiv t \bmod n$

Exercises

1. Find all generators of $(\mathbb{Z}_8, +)$
   (a) Find all proper subgroups of $(\mathbb{Z}_8, +)$ and list their generators
   (b) Draw a subgroup diagram of $(\mathbb{Z}_8, +)$
2. Find all generators of $(\mathbb{Z}_9, +)$
   (a) Find all proper subgroups of $(\mathbb{Z}_9, +)$ and list their generators
   (b) Draw a subgroup diagram of $(\mathbb{Z}_9, +)$
3. Find all generators of $(\mathbb{Z}_7, +)$
   (a) Find all proper subgroups of $(\mathbb{Z}_7, +)$ and list their generators
   (b) Draw a subgroup diagram of $(\mathbb{Z}_7, +)$
4. Find all generators of $(\mathbb{Z}_{10}, +)$
   (a) Find all proper subgroups of $(\mathbb{Z}_{10}, +)$ and list their generators
   (b) Draw a subgroup diagram of $(\mathbb{Z}_{10}, +)$
5. Find all generators of $(\mathbb{Z}_{12}, +)$
   (a) Find all proper subgroups of $(\mathbb{Z}_{12}, +)$ and list their generators
   (b) Draw a subgroup diagram of $(\mathbb{Z}_{12}, +)$
6. Find all generators of $(U_5, \cdot)$
   (a) Find all proper subgroups of $(U_5, \cdot)$ and list their generators
   (b) Draw a subgroup diagram of $(U_5, \cdot)$
7. Find all generators of $(U_7, \cdot)$
   (a) Find all proper subgroups of $(U_7, \cdot)$ and list their generators
   (b) Draw a subgroup diagram of $(U_7, \cdot)$
8. Find all generators of $(U_{11}, \cdot)$
   (a) Find all proper subgroups of $(U_{11}, \cdot)$ and list their generators
   (b) Draw a subgroup diagram of $(U_{11}, \cdot)$

9. Find all generators of $(\mathbb{Z}_2 \times \mathbb{Z}_5, +)$
   (a) Find all proper subgroups of $(\mathbb{Z}_2 \times \mathbb{Z}_5, +)$ and list their generators
   (b) Draw a subgroup diagram of $(\mathbb{Z}_2 \times \mathbb{Z}_5, +)$
10. Find all generators of $(\mathbb{Z}_3 \times \mathbb{Z}_3, +)$
    (a) Find all proper subgroups of $(\mathbb{Z}_3 \times \mathbb{Z}_3, +)$ and list their generators
    (b) Draw a subgroup diagram of $(\mathbb{Z}_3 \times \mathbb{Z}_3, +)$
11. Define an isomorphism between $(\mathbb{Z}_{10}, +)$ and $(U_{11}, \cdot)$
12. Define an isomorphism between $(\mathbb{Z}_{10}, +)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_5, +)$
13. Define an isomorphism between $(\mathbb{Z}_9, +)$ and $(\mathbb{Z}_3 \times \mathbb{Z}_3, +)$
14. Find all generators of $(\mathbb{Z}, +)$
    (a) List 4 subgroups of $(\mathbb{Z}, +)$ and list their generators
    (b) Characterize all subgroups of $(\mathbb{Z}, +)$
15. Find all generators of $(\{2^n : n \in \mathbb{Z}\}, \cdot)$
    (a) List 4 subgroups of $(\{2^n : n \in \mathbb{Z}\}, \cdot)$ and list their generators
    (b) Characterize all subgroups of $(\{2^n : n \in \mathbb{Z}\}, \cdot)$
16. Define an isomorphism between $(\mathbb{Z}, +)$ and $(\{2^n : n \in \mathbb{Z}\}, \cdot)$