MTH 4441 Homework Exercises Set #7a - Permutations Part #2 - Solutions

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In Exercises 1-4, decompose the given permutations into the "product" of disjoint cycles. (Remember - the "product" of the cycles is really function composition, so we proceed from right to left.)

Remark: The procedure for this is rather straightforward:

Start with the element "1" and follow this element through repeated applications of the permutation until you return to the element "1." The elements that are produced, as well as the order in which these elements appear, define one cycle.

Select the first element of the set that does not appear in the previous cycle. Follow this element through repeated applications of the permutation until you return to the element. The elements that are produced, as well as the order in which these elements appear, define the next cycle.

Continue with this process, until you have exhausted all elements.

1. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}$

   Observe: $\sigma(1) = 5$

   $\sigma^2(1) = \sigma(\sigma(1)) = \sigma(5) = 2$

   $\sigma^3(1) = \sigma(\sigma^2(1)) = \sigma(2) = 1$

   Thus, $(1, 5, 2)$ is one cycle.

   The element "3" is the first element that does not appear in the previous cycle. So apply σ repititively to "3."

   $\sigma(3) = 3$

   i.e. "3" is not moved by the permutation, so "3" is not part of **any** cycle.

   We move to "4"

   $\sigma(4) = 6$

   $\sigma^2(4) = \sigma(\sigma(4)) = \sigma(6) = 4$

   Thus, $(4, 6)$ is the next cycle.

   This exhausts all elements.

i.e., $\sigma = (4, 6)(1, 5, 2) = (1, 5, 2)(4, 6)$

   Note: Since the cycles are **disjoint,** the **order** of the cycles is **arbitrary,** because **disjoint cycles commute.**

2. $\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$

Observe: $\phi(1) = 3$

$\phi^2(1) = \phi(\phi(1)) = \phi(3) = 4$

$\phi^3(1) = \phi\left(\phi^2(1)\right) = \phi(4) = 5$

$\phi^4(1) = \phi\left(\phi^3(1)\right) = \phi(5) = 6$

$\phi^5(1) = \phi\left(\phi^4(1)\right) = \phi(6) = 2$

$\phi^6(1) = \phi\left(\phi^5(1)\right) = \phi(2) = 1$

Thus, $(1, 3, 4, 5, 6, 2)$ is one cycle.

This exhausts all elements.

i.e., $\phi = (1, 3, 4, 5, 6, 2)$

3. $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}$

Observe: $\tau(1) = 5$

$\tau^2(1) = \tau(\tau(1)) = \tau(5) = 8$

$\tau^3(1) = \tau\left(\tau^2(1)\right) = \tau(8) = 7$

$\tau^4(1) = \tau\left(\tau^3(1)\right) = \tau(7) = 1$

Thus, $(1, 5, 8, 7)$ is the first cycle.

The element "2" is the first element that does not appear in the previous cycle. So apply σ repititively to "2."

$\tau(2) = 6$

$\tau^2(2) = \tau(\tau(2)) = \tau(6) = 3$

$\tau^3(2) = \tau\left(\tau^2(2)\right) = \tau(3) = 2$

Thus, $(2, 6, 3)$ is the next cycle.

Finally, we move to "4." But "4" is not moved by the permutation, so "4" is not part of **any** cycle.

This exhausts all elements.

i.e., $\tau = (1, 5, 8, 7)(2, 6, 3) = (2, 6, 3)(1, 5, 8, 7)$

Note: Since the cycles are **disjoint,** the **order** of the cycles is **arbitrary,** because **disjoint cycles commute.**

4. $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 1 & 4 & 6 & 8 & 7 \end{pmatrix}$

Observe: $\mu(1) = 2$

$\mu^2(1) = \mu(\mu(1)) = \mu(2) = 3$

$\mu^3(1) = \mu(\mu^2(1)) = \mu(3) = 5$

$\mu^4(1) = \mu(\mu^3(1)) = \mu(5) = 4$

$\mu^5(1) = \mu(\mu^4(1)) = \mu(4) = 1$

Thus, $(1, 2, 3, 5, 4)$ is the first cycle.

The element "6" is the first element that does not appear in the previous cycle, but "6" is not moved by the permutation, so "6" is not part of **any** cycle.

We move to "7." So apply μ repititively to "7."

$\mu(7) = 8$

$\mu^2(7) = \mu(\mu(7)) = \mu(8) = 7$

Thus, $(7, 8)$ is the final cycle.

This exhausts all elements.

i.e., $\mu = (1, 2, 3, 5, 4)(7, 8) = (7, 8)(1, 2, 3, 5, 4)$

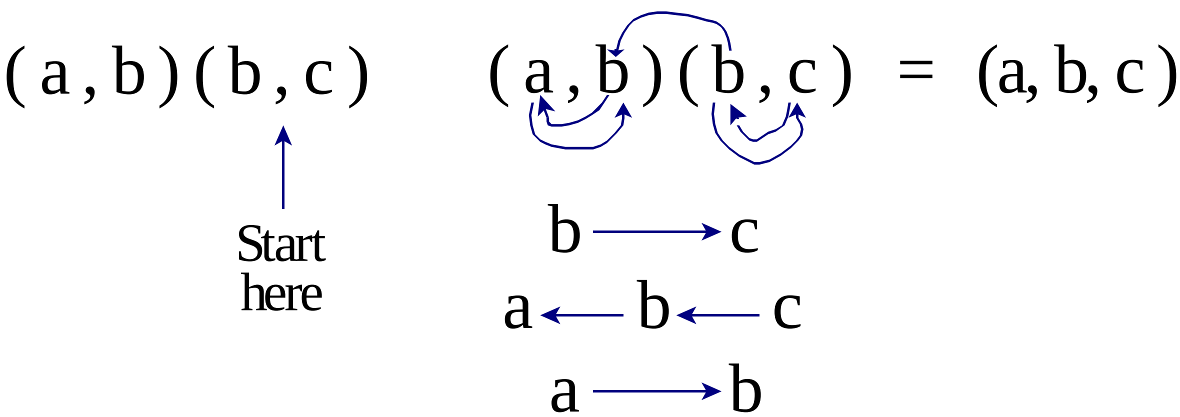
Note: Since the cycles are **disjoint,** the **order** of the cycles is **arbitrary,** because **disjoint cycles commute.**

In Exercises 5-11, decompose the given permutations into the "product" of transpositions. (Remember - the "product" of the cycles is really function composition, so we proceed from right to left.)

Remark: It would probably be worth our time, before beginning this section, to develop a systematic method of decomposing cycles into the "product" of transpositions.

First, we consider the "3-cycle" (a, b, c).

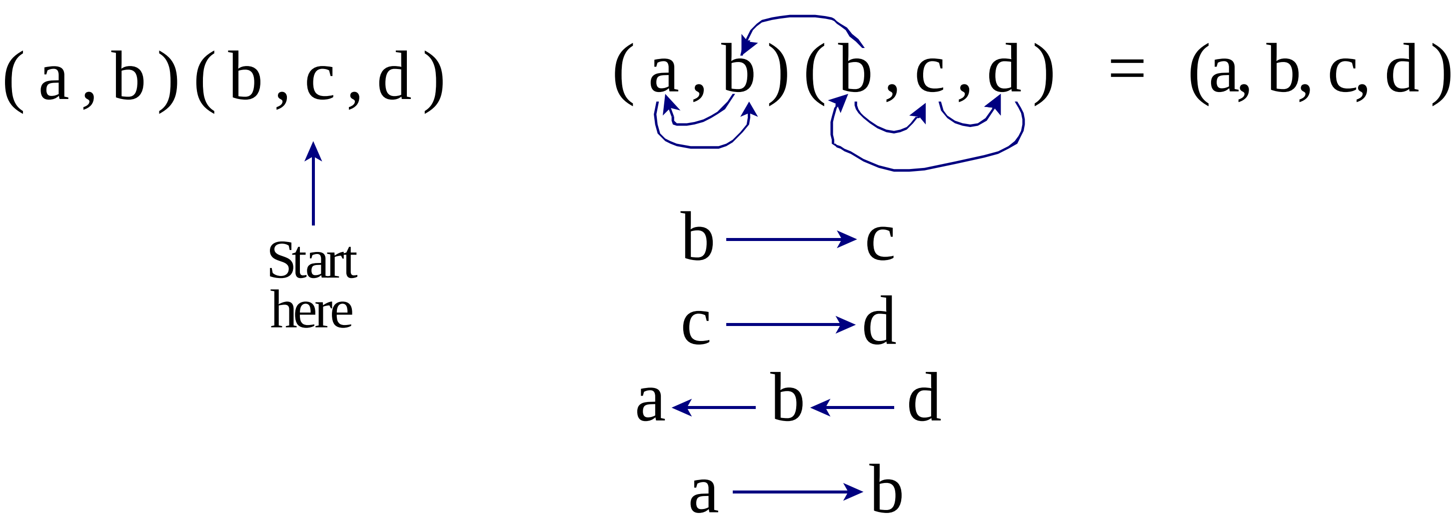
Observe (below) that $(a, b, c) = (a, b)(b, c)$ (Keep in mind that this "product" is actually function composition, and as such, is performed from **right to left.)**



The Point: The 3-cycle (a, b, c) can be expressed as $(a, b, c) = (a, b)(b, c)$.

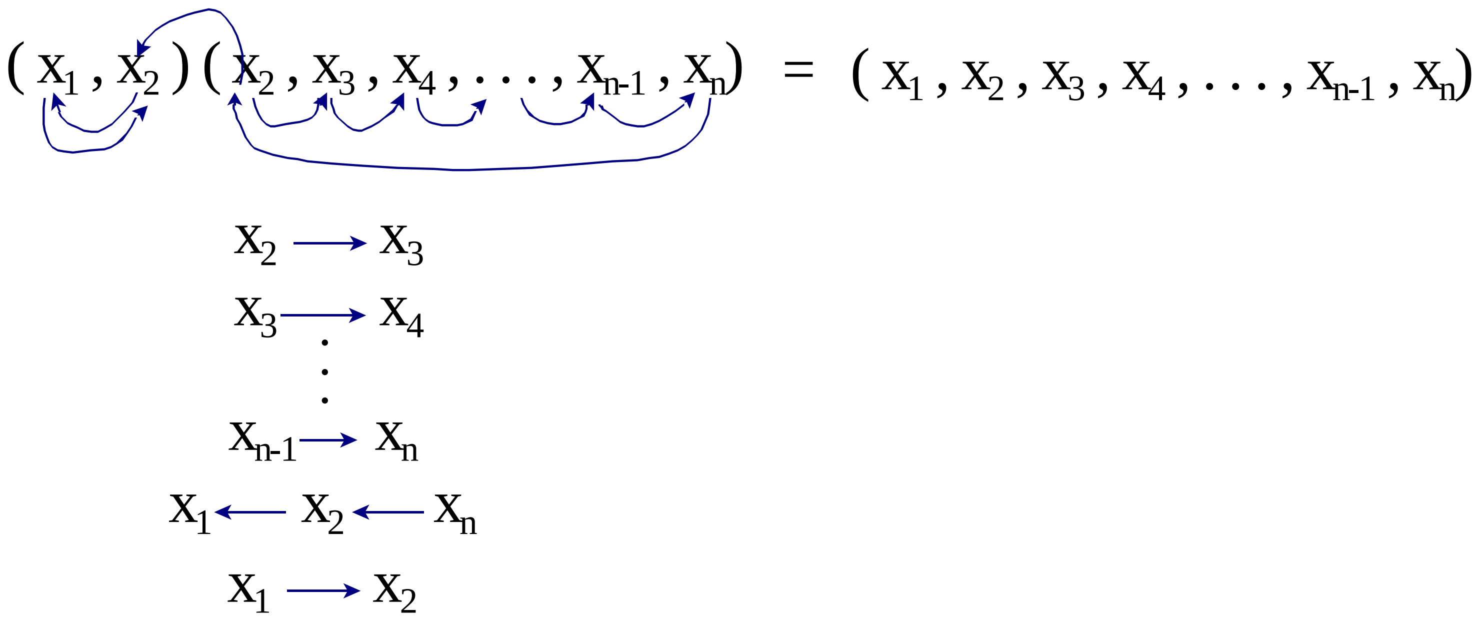
Next, we consider the "4-cycle" (a, b, c, d)

Observe (below) that $(a, b, c, d) = (a, b)(b, c, d)$ (Keep in mind that this "product" is actually function composition, and as such, is performed from **right to left.)**



The Point: The 4-cycle (a, b, c, d) can be expressed as $(a, b, c, d) = (a, b)(b, c, d)$.

In General: The n-cycle $(x_1, x_2, x_3, x_4, \ldots, x_{n-1}, x_n)$ can be expressed as $(x_1, x_2, x_3, x_4, \ldots, x_{n-1}, x_n) = (x_1, x_2)\,(x_2, x_3, x_4, \ldots, x_{n-1}, x_n)$, as shown below:



The Point: $(x_1, x_2, x_3, x_4, \ldots, x_{n-1}, x_n) = (x_1, x_2)\,(x_2, x_3, x_4, \ldots, x_{n-1}, x_n)$

5. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}$

From Exercise 1, $\sigma = (4,6)\,(1,5,2) = (1,5,2)\,(4,6)$

We must express the cycle $(1,5,2)$ as the product of transpositions.

Since $(1,5,2) = (1,5)\,(5,2)$, we can express σ as follows:

$$\sigma = (4,6)\underbrace{(1,5)\,(5,2)}_{=(1,5,2)} = \underbrace{(1,5)\,(5,2)}_{=(1,5,2)}(4,6)$$

i.e., $\sigma = (4,6)\,(1,5)\,(5,2)$ or $\sigma = (1,5)\,(5,2)\,(4,6)$

Generally speaking, the order of the transpositions can not be changed arbitrarily, since the transpositions are NOT disjoint.

6. $\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$

From Exercise 2, $\phi = (1,3,4,5,6,2)$

We must express the cycle $(1,3,4,5,6,2)$ as the product of transpositions.

Observe: $(1,3,4,5,6,2) = (1,3)\,(3,4,5,6,2) = (1,3)\,\underbrace{(3,4)\,(4,5,6,2)}_{=(3,4,5,6,2)} = (1,3)\,(3,4)\,\underbrace{(4,5)\,(5,6,2)}_{=(4,5,6,2)}$

$$= (1,3)\,(3,4)\,(4,5)\,\underbrace{(5,6)\,(6,2)}_{=(5,6,2)}$$

i.e. $\phi = (1,3)\,(3,4)\,(4,5)\,(5,6)\,(6,2)$

Generally speaking, the order of the transpositions can not be changed arbitrarily, since the transpositions are NOT disjoint.

7. $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}$

From Exercise 3, $\tau = (1,5,8,7)\,(2,6,3) = (2,6,3)\,(1,5,8,7)$

We must express the cycles $(1,5,8,7)$ and $(2,6,3)$ as the products of transpositions.

$(1,5,8,7) = (1,5)\,(5,8,7) = (1,5)\,\underbrace{(5,8)\,(8,7)}_{=(5,8,7)}$

i.e., $(1,5,8,7) = (1,5)\,(5,8)\,(8,7)$

also: $(2,6,3) = (2,6)\,(6,3)$

Thus, $\tau = (1,5,8,7)\,(2,6,3) = \underbrace{(1,5)\,(5,8)\,(8,7)}_{=(1,5,8,7)}\underbrace{(2,6)\,(6,3)}_{=(2,6,3)}$

or: $\tau = (2,6,3)\,(1,5,8,7) = \underbrace{(2,6)\,(6,3)}_{=(2,6,3)}\underbrace{(1,5)\,(5,8)\,(8,7)}_{=(1,5,8,7)}$

i.e., $\tau = (1,5)\,(5,8)\,(8,7)\,(2,6)\,(6,3)$ or $\tau = (2,6)\,(6,3)\,(1,5)\,(5,8)\,(8,7)$

Generally speaking, the order of the transpositions can not be changed arbitrarily, since the transpositions are NOT disjoint.

8. $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 1 & 4 & 6 & 8 & 7 \end{pmatrix}$

From Exercise 4, $\mu = (1,2,3,5,4)\,(7,8) = (7,8)\,(1,2,3,5,4)$

We must express the cycle $(1,2,3,5,4)$ as the product of transpositions.

Observe: $(1,2,3,5,4) = (1,2)\,(2,3,5,4) = (1,2)\,\underbrace{(2,3)\,(3,5,4)}_{=(2,3,5,4)} = (1,2)\,(2,3)\,\underbrace{(3,5)\,(5,4)}_{=(3,5,4)}$

i.e., $(1,2,3,5,4) = (1,2)\,(2,3)\,(3,5)\,(5,4)$

Thus, $\mu = (1,2,3,5,4)\,(7,8) = \underbrace{(1,2)\,(2,3)\,(3,5)\,(5,4)}_{=(1,2,3,5,4)}\,(7,8)$

or: $\mu = (7,8)\,(1,2,3,5,4) = (7,8)\,\underbrace{(1,2)\,(2,3)\,(3,5)\,(5,4)}_{=(1,2,3,5,4)}$

i.e., $\mu = (1,2)\,(2,3)\,(3,5)\,(5,4)\,(7,8)$ or $\mu = (7,8)\,(1,2)\,(2,3)\,(3,5)\,(5,4)$

Generally speaking, the order of the transpositions can not be changed arbitrarily, since the transpositions are NOT disjoint.

9. $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$

First, we must express ω as the "product" of disjoint cycles.

Observe: $\omega(1) = 8$

$\omega^2(1) = \omega(\omega(1)) = \omega(8) = 1$

Thus, $(1, 8)$ is the first cycle.

The element "2" is the first element that does not appear in the previous cycle, but "2" is not moved by the permutation, so "2" is not part of **any** cycle.

We move to "3." So apply ω repititively to "3."

$\omega(3) = 6$

$\omega^2(3) = \omega(\omega(3)) = \omega(6) = 4$

$\omega^3(3) = \omega\left(\omega^2(3)\right) = \omega(4) = 3$

Thus, $(3, 6, 4)$ is the next cycle.

The element "5" is the first element that does not appear in the previous cycle, so we apply ω repititively to "5."

$\omega(5) = 7$

$\omega^2(5) = \omega(\omega(5)) = \omega(7) = 5$

Thus, $(5, 7)$ is the last cycle.

This exhausts all elements.

i.e., $\omega = (1, 8)(3, 6, 4)(5, 7)$ (expressed in any order).

Next, we must express $(3, 6, 4)$ as the "product" of transpositions.

$(3, 6, 4) = (3, 6)(6, 4)$

Thus ω can be expressed as the "product" of transpositions in any of the following 6 ways:

$$\omega = (1,8)\underbrace{(3,6)(6,4)}_{=(3,6,4)}(5,7) = (5,7)\underbrace{(3,6)(6,4)}_{=(3,6,4)}(1,8) = (5,7)(1,8)\underbrace{(3,6)(6,4)}_{=(3,6,4)} = (1,8)(5,7)\underbrace{(3,6)(6,4)}_{=(3,6,4)}$$

$$= \underbrace{(3,6)(6,4)}_{=(3,6,4)}(1,8)(5,7) = \underbrace{(3,6)(6,4)}_{=(3,6,4)}(5,7)(1,8)$$

10. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$

First, we must express π as the "product" of disjoint cycles.

Observe: $\pi(1) = 3$

$\pi^2(1) = \pi(\pi(1)) = \pi(3) = 4$

$\pi^3(1) = \pi(\pi^2(1)) = \pi(4) = 1$

Thus, $(1,3,4)$ is the first cycle.

The element "2" is the first element that does not appear in the previous cycle, so we apply π repititively to "2."

$\pi(2) = 6$

$\pi^2(2) = \pi(\pi(2)) = \pi(6) = 2$

Thus, $(2,6)$ is the next cycle.

The element "5" is the first element that does not appear in the previous cycle, so we apply π repititively to "5."

$\pi(5) = 8$

$\pi^2(5) = \pi(\pi(5)) = \pi(8) = 7$

$\pi^3(5) = \pi(\pi^2(5)) = \pi(7) = 5$

Thus, $(5,8,7)$ is the last cycle.

This exhausts all elements.

Thus, $\pi = (1,3,4)(2,6)(5,8,7)$

We now decompose the cycles $(1,3,4)$ and $(5,8,7)$ into transpositions.

$(1,3,4) = (1,3)(3,4)$

$(5,8,7) = (5,8)(8,7)$

Thus π can be expressed as the "product" of transpositions in any of the following 6 ways:

$$\pi = \underbrace{(1,3)(3,4)}_{=(1,3,4)}(2,6)\underbrace{(5,8)(8,7)}_{=(5,8,7)} = \underbrace{(1,3)(3,4)}_{=(1,3,4)}\underbrace{(5,8)(8,7)}_{=(5,8,7)}(2,6) = \underbrace{(5,8)(8,7)}_{=(5,8,7)}\underbrace{(1,3)(3,4)}_{=(1,3,4)}(2,6)$$

$$= \underbrace{(5,8)(8,7)}_{=(5,8,7)}(2,6)\underbrace{(1,3)(3,4)}_{=(1,3,4)} = (2,6)\underbrace{(5,8)(8,7)}_{=(5,8,7)}\underbrace{(1,3)(3,4)}_{=(1,3,4)} = (2,6)\underbrace{(1,3)(3,4)}_{=(1,3,4)}\underbrace{(5,8)(8,7)}_{=(5,8,7)}$$

11. $\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$

First, we must express π as the "product" of disjoint cycles.

Observe: $\lambda(1) = 3$

$\lambda^2(1) = \lambda(\lambda(1)) = \lambda(3) = 4$

$\lambda^3(1) = \lambda\left(\lambda^2(1)\right) = \lambda(4) = 7$

$\lambda^4(1) = \lambda\left(\lambda^3(1)\right) = \lambda(7) = 8$

$\lambda^5(1) = \lambda\left(\lambda^4(1)\right) = \lambda(8) = 6$

$\lambda^6(1) = \lambda\left(\lambda^5(1)\right) = \lambda(6) = 5$

$\lambda^7(1) = \lambda\left(\lambda^6(1)\right) = \lambda(5) = 2$

$\lambda^8(1) = \lambda\left(\lambda^7(1)\right) = \lambda(2) = 1$

Since this exhausts all of the elements, this is the only cycle: $(1,3,4,7,8,6,5,2)$.

Thus, $\lambda = (1,3,4,7,8,6,5,2)$

We now decompose $\lambda = (1,3,4,7,8,6,5,2)$ into transpositions.

$$\lambda = (1,3,4,7,8,6,5,2) = (1,3)(3,4,7,8,6,5,2) = (1,3)\underbrace{(3,4)(4,7,8,6,5,2)}_{=(3,4,7,8,6,5,2)}$$

$$= (1,3)(3,4)\underbrace{(4,7)(7,8,6,5,2)}_{=(4,7,8,6,5,2)} = (1,3)(3,4)(4,7)\underbrace{(7,8)(8,6,5,2)}_{=(7,8,6,5,2)}$$

$$= (1,3)(3,4)(4,7)(7,8)\underbrace{(8,6)(6,5,2)}_{=(8,6,5,2)} = (1,3)(3,4)(4,7)(7,8)(8,6)\underbrace{(6,5)(5,2)}_{=(6,5,2)}$$

$$\boxed{\lambda = (1,3)(3,4)(4,7)(7,8)(8,6)(6,5)(5,2)}$$